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Devoted to the interests of mathematics in Elementary and Secondary Schools

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THE MATHEMATICS TEACHER

Volume XXIV Number 5

Edited by William David Reeve

The National Council of Teachers of Mathematics, Guest and Servant of the Detroit Public Schools

By E. L. Miller, *Assistant Superintendent of Detroit Public Schools, in charge of High Schools, and formerly president of the National Council of Teachers of English.*

I have been asked to say something to you about the teaching of English. I will undertake to do this as painlessly as possible, and to secure that end I will convey what I have to say to you in the form of an apologue:

Among the treasures which have recently been unearthed on the banks of the Euphrates is a Babylonian wall on which is inscribed in cuneiform characters the following autobiographical fragment from the chisel of an unknown pedagogue:

"It was September of the year 1330 B.C. I stood trembling before Dr. Bulbul, superintendent of the schools of Babylon. 'So,' said he, 'thou desirest a job in a high school?' 'Yea, verily,' said I. 'What dost thou wish to teach,' said he. 'Babylonian,' said I. 'Indeed,' said he contemptuously. 'Knowest thou not that any fool can teach Babylonian?' 'That's the reason,' said I. 'Canst thou teach aught else?' said he. 'Yes,' said I modestly but unveraciously. 'I can teach Chinese, Sanskrit, Arabic, Egyptian, Indo-European, mathe-

matics, history, astronomy, and the art of baking bricks.' Whereupon he interrupted me with boisterous but not ungenial cachinations, saying: 'Thou wilt report for duty at the Nebuchadnezzar High School on Monday at 4.00 A.M.'

"The system of schools into which I was thus inducted was cursed with a course in Babylonian which was strictly up-to-date. It consisted of a list of Babylonian classics, some of which had been selected by the teachers because they liked them but most by Dr. Bulbul because he had read their titles in a list of the hundred best walls compiled by Tiglath-Pelouzar, the imperial biologist. Of those titles I can remember only two—"The Large Granite Mug" and "The Shoemaker Resold." There was in the curriculum no vestige of those survivals of a barbaric past which are known to archeologists as spelling, punctuation, capitalization, or the other essentials of good usage.

"In this fair garden of letters I luxuriated for some years, happy and beloved by my pupils.

"It so happened in those days that one Lithocrates, a man of low origin but great cunning, had acquired in trade much gold and unpopularity. Now, waxing old and fearing the wrath of the gods or desiring to corrupt the minds of the people, he founded at Babylon an university, bringing thither at great expense wondrous football players from the four quarters of the globe, among them from Palestine one Samson, who later attained an evil notoriety. But that is another story. He brought also from the Troad a young professor of Babylonian named Homer, who entertained the to-me-novel idea that graduates of high schools should be able to speak and write Babylonian with fluency and precision. As those among my disciples who repaired to his classes could do neither, he made complaint of me to Dr. Bulbul, saying that I knew not how to teach Babylonian. Now, be it understood, Babylonian meant to me literature, to Professor Homer composition, and to Dr. Bulbul everything in general and nothing in particular. Whence, as commonly happens when three persons enter into disputation about a question which none of them understands and a clear conception of which dependeth on the definition of a word to which each attacheth a different meaning, arose acrimonious strife both written and oral; and I retired from the fray humbled and chastened even as the grain of millet is ground betwixt the upper and nether millstones.

"After long meditation upon these matters I did finally perceive

as in a glass darkly that the curriculum required me to teach literature while Professor Homer, who was himself inordinately fond of scribbling, thought only of composition. Therefore, being a man of peace, I tried to teach both, and, as an end to that means, required that each of my pupils annually fashion and bake one brick. The result was that my own pleasures were seriously curtailed. I often sat up all night with my mallet and my chisel working upon the compositions of my disciples in the effort to bring them down to the level required by the professor. Strange to say, my labors were not appreciated. My fellow teachers made bitter moan, saying that it was contrary to union ethics to work overtime. The children rebelled because Babylonian was no longer a snap. Certain parents insinuated that I owned stock in a brick-yard. One mother even went so far as to inform Mr. Fortinbras, my principal, that I was a crank on the subject of Babylonian. In his alarm that astute but timid functionary rebuked me, saying: 'Each one has their own rights. The progenitors of Hilpah and Shalum wax wroth. Do not give so much work to her and he.' The uncertainty of his syntax need occasion no surprise. I note increasing laxity of language in all grades of Babylonian society since the tower episode.

"When these things came to the ears of Dr. Bulbul, he sent for me and spake thus: 'What the children need is not spelling but inspiration. As for punctuation, by hen, oddsbodykins, and likewise pish! When there is a break in the thoughts of thy disciples, let them put in a dash.' 'If that is thy system,' said I, 'thy compositions must resemble an untranslated Marconi cablegram. 'Moreover,' I added, 'if I were to print my opinion of thy system, I could express my sentiments only by a series of dashes.'

"After which I abandoned the profession of pedagogy, formed a connection with a brick-maker of Babylon, and went up and down in the land selling primers to school boards, an occupation which proved to be more lucrative than my previous one."

The moral of this fable, I take it, is that the teaching of the vernacular presents problems that are baffling by reason of their complexity. As somebody once said somewhere of one of Emerson's lectures, it begins everywhere and ends nowhere. The most illuminating remark that I know concerning its real nature is to be found among the utterances of a pedagogical authority not as yet widely recognized as such, to wit, Dogberry in *Much Ado about Nothing*,

Act II, Scene 3. He says: "To be a well-favored man is the gift of fortune; but to read and write comes by nature." In other words, if as English teachers you wish to turn out from your schools a finished product, select for your pupils those with cultured grandparents.

All of this is important to mathematicians because, if English is to be taught at all, it must be taught, as it is used, in all classes. Among the best media I know for teaching English are algebra and geometry. The ability and habit of exact and correct expression are developed in pupils by one of our mathematics teachers more successfully than by most English teachers. Indeed I am inclined to believe that, for most pupils, the English training they get from mathematics is more important than the mathematical. This English training all of them will use every day of their lives. Most of them will use their mathematics almost never.

For the last twenty years, you have been struggling to produce a hodge-podge of arithmetic, geometry, algebra, which you have combined under the name of general mathematics. In English we have been trying, as the old Babylonian schoolmaster tried, to unscramble a similar mess. We find that, when literature and composition are combined, usually no composition is taught. The reason is that, while literature teaching is one of the most delightful of pastimes, composition teaching is too strenuous an occupation for any except the strongest souls. If some composition does get taught along with literature, this sort of thing often happens. Willie gets 100 in literature and 40 in composition and passes with an average of 70 without being able to tell the difference between a sentence and a growl. Mixed English classes are like mixed trains—slow and jerky. Separate classes are like the limited; they get somewhere fast. A mixed class resembles the dirty hovel of a peasant, where pig, parlor, cookstove, and bed are mingled in barbarous confusion. Separated classes are like the decent rooms in a civilized American home, where cookery, gastronomical enjoyment, conversation, study, and slumber are segregated. Even in such homes it sometimes happens, however, that a lazy housewife will violate the unities of domesticity, and even so, when you have a well-organized course in English, only eternal vigilance will prevent some lazy teacher from trying to substitute grammar for composition.

I congratulate you because you, as mathematics teachers, are free from these troubles.

The Training of Teachers of Mathematics With Special Reference to the Relation of Mathematics to Modern Thought[1]

By W. D. CAIRNS, *Oberlin College*

THE NATIONAL COUNCIL has in its recent programs been considering current problems, topics in the teaching of mathematics, the revision of courses in algebra and in geometry, and various other topics, and has incorporated these in the Council's notable series of Yearbooks. This consideration is wise and timely and it makes the Yearbooks most valuable for the present and prospective teachers of mathematics. I want to develop the thesis that along with all of this inspiring work much remains to be done in the training of teachers in order that we may help them to extend their horizons.

First of all I am a hearty supporter of training courses and training methods which will give to the inexperienced teachers a proper knowledge and technique in the teaching of mathematics. A generation ago it sufficed that a teacher should be a born mathematician and that he should be an expert drill master, and I have known many good teachers of mathematics of that era to deride the new courses in methods. That day has long since passed and one needs to justify these courses neither in such an assemblage as this nor to the superintendents and principals of secondary schools.

In the second place it is at least equally important that there should be established courses in the subject-matter in secondary and collegiate mathematics which shall be adequate for the needs of our teachers. For a number of years it has been very clear that the methods courses have been over-developed at the expense of content courses. This might well be a beautiful topic for a lively debate, but I waive the question as not pertinent to the main topic.

I have little doubt that my third point is today the most important of the three, viz., our teachers of mathematics—and these teachers are not unique in this respect—our teachers of mathematics live too entirely within the domain in which they are teaching. I do

[1] An address given at the annual banquet of the National Council of Teachers of Mathematics, Detroit, Michigan, Feb. 21, 1931.

not even mean to say "they"; I mean we ourselves. We are limiting ourselves for the most part too much to the special courses in hand, and failing both to educate ourselves as individuals in the broader fields of human thought and culture and to study the bearings of these other fields upon our own.

There are of course two principles underlying our work which are more or less complementary: we must develop our courses as pure mathematics with the proper mastery of technique, and we must, along with this, exhibit to our pupils as wide a view of the applications of mathematics as is consistent with the limitations of time and of the maturity of the students. The corresponding statement may be made with respect to the training of teachers of mathematics. Doctor K. T. Compton in his inaugural address as president of the Massachusetts Institute of Technology in June, 1930, emphasized the necessity of greater emphasis upon the fundamental sciences. Speaking especially of engineering, he said that as it (engineering) has developed to greater and greater complexities, it has become increasingly impossible to hope to train men in those exact processes of thought or manipulation for which they will later be called upon, and that therefore a broad and thorough training in fundamental principles gives much greater power than a training in details which may seldom be encountered in practice. This emphasis by Doctor Compton on the first of the two principles I have mentioned is very noteworthy.

On the other hand a number of workers have been pushing the knowledge of our teachers into other fields not previously known to a part at least. I recall, for example, the paper by Miss Gugle in the *Third Yearbook* on "Dynamic Symmetry." I note too in the same Yearbook the paper by Mr. Schlauch on "Mathematics as an Interpreter of Life," with his reaching out to social efficiency and citizenship, a general understanding of, and an insight into, economic, social, and cosmic forces. I quote one passage from him: "If it (mathematics) is properly taught, with inspiring sidelights and comments by a live teacher, there should emerge in the mind of the student a concept of an ordered, lawful universe in which all phenomena yield to quantitative investigation." And in a book entitled *Mathematics for Junior High School Teachers* just published by Dr. Schaaf of the Preparatory High School, College of the City of New York, I am glad to see references correlating mathematics with science, economics, history, philosophy, and logic. A member of our staff at

Oberlin College has just made an attractive list of brief readings in the applications of mathematics in ten other fields, for the use of her freshman students. Professors Hedrick, Judd, and David Eugene Smith have among others shown in great plenty the existence of mathematical ideas in the world of today and the bearing of mathematical training on non-mathematical subjects. As Professor Judd said in his address at the Dallas meeting in February, 1927, "The individual who has had experience with number is no longer capable of returning to the level of loose, inexact thinking that characterized his earlier methods of viewing the world." The implication which occurs to me here is the possibility of carrying these fundamental values over into other fields, the responsibility of the teacher to extend his knowledge of these ramifications of what is essentially mathematics even into remote fields, and our responsibility for its utilization in our teaching.

What I am seeking to emphasize is even a larger concept and a larger demand. While no one would maintain that mathematics comprises within its scope all of the universe and all forms of thinking and of general human experience, nevertheless the present age is learning that the notions which we call mathematical extend far more widely over these fields than has been popularly supposed. Let me illustrate this by several examples which may very well be known to you in part or wholly.

1. We glory in mathematics as the very special domain in which we may prove statements with an assurance of certainty. Notable contributions to the science of formal logic have been made in the past century by mathematicians while they were engaged in placing the foundations of advanced mathematical theory on a strictly logical basis. More recently, however, namely in 1908, the Dutch mathematician Brouwer denied the universal validity of the Law of the Excluded Middle, viz: the logical statement that of the two propositions, "A is B" and "A is not B," one must always hold, whatever A and B may be. Since this denial removes one of the three postulates of Aristotle which are the basis of all reasoning, this precipitated a controversy among logicians that still continues. The most striking consequence of all this is, as stated by Professor E. T. Bell in a jovial tract called "Debunking Science" (*University of Washington Chapbooks*, 1930), that there is a sharp disagreement among competent men as to what can be proved and what cannot be proved, and that the

dispute is between those men of science who are most competent, in the common opinion of experts, to judge the issue. It is important that however large the inertia of our own intellectual habits, the oncoming teacher of mathematics should know that mathematics reaches with a powerful grasp into the domain of logic and that the very foundations of these subjects, or rather our formulation of these logical structures, are still insecure. An account of this may be found in an article by Dresden in the *Bulletin of the American Mathematical Society* for 1924.

2. Mathematics is proving increasingly successful in its treatment of the complicated phenomena of economic theory. Scholars in economics, as in any other science, seek to express the laws of the phenomena in their field exactly, fully, clearly, to analyze the dependence of one quantity on other economic factors, for example, to find what contributing causes determine the market price of a commodity, what the effect of monopoly in a market is, and many other involved questions. A reading of articles by Professor C. C. Morris in the *American Mathematical Monthly* for 1924, and by Professor G. C. Evans in the same journal for 1925, will bring to any thinking student of mathematics a great satisfaction resulting from the knowledge that mathematics is contributing to the effective study of such questions and giving a partial solution to these. It lies with us to fortify our pupils so that they may be ready with the technique to work along these lines, not to speak of the present-day trend toward statistical and actuarial applications.

3. The field of biology seems to the most of us to consist of dry laws of a physiological nature, with laboratory hours devoted to the dissection of flowers, frogs, and fishworms. Yet we and our pupils may learn of a surprising amount of aid which mathematics gives in this field. I recall that a number of years ago Professor Horatio B. Williams of the College of Physicians and Surgeons was engaged in a study of the transmission of stimuli along nerves. In order to have at hand an apparatus which might give a close parallel to the properties of nerves, he constructed, by mathematical calculation, an electric cable which had precisely the same electrical characteristics as were possessed by the nerve which he was studying. In particular it transmitted an impulse along the cable with just the same velocity as did the nerve. This formed an important basis for his further investigation in that special line of research. Happily he possessed the mathematical training requisite for his task.

4. Mr. Betz referred last night to the phenomena of a cosmic order which are forcing themselves on the attention of all educated and thinking minds. The latest of these to be announced from Mount Wilson Observatory is what Doctor F. H. Sears calls a cosmic sunset, in the case of a group of stars in the constellation Cygnus which is just on the edge of an enormous cloud of dust particles. Now how does he know all this? It is perfectly simple and within the understanding of any of the teachers whom we are training. The astronomer is used to listing stars according to their colors; many are white, but others emit light which has a bluish tinge and still others give a reddish light. More exactly stated, spectrum analysis has enabled astronomers to classify many groups of stars and to establish the percentages that have an excess of blue or of red light or intermediate colors. Now the light coming from this group of 750 stars shows an excess of red light, just as do the sun's rays at sunrise or sunset. It means then that some cloud-like disturbance intervenes between these stars and the earth. All this, mind you, involves merely such mathematics and physics as any one can acquire in a sophomore college course in physics. The same sort of analysis shows that this disturbance of the light cannot arise from a dark mass of gases, for it would require so large a body of gas that this would itself form a star, emit light, etc. Moreover, the obstruction cannot be made up of bodies like our asteroids, for these would actually intercept the light instead of causing it to deviate and to become redder. The only explanation is that this must be a cloud of dust particles which are smaller in their linear dimensions than the wave length of light. Perfectly logical, a beautiful example of the method of indirect proof, and yet we are talking about what happened to certain rays of light several hundreds of years ago as they passed that region of space. Our information is belated but it is much more trustworthy than the record of events on the earth several hundred years ago.

5. Are you ever asked about the new physics of the atom? Do your pupils or your young fellow-teachers want to know what it is all about, what kind of mathematics is involved in its study, what Bohr's model of the atom is, what it accomplished in explaining facts not previously explained, why it has been largely discarded, etc.? I have been dreadfully embarrassed by such questions and have been studying the newer developments this winter so as to keep up with that part of the procession.

The study of the atom has been advancing since 1900 at an as-

tonishing rate, the speed with which several complete systems have been developed, approved by scientists, and then discarded, being almost a scandal in the history of physics in the twentieth century. Professor Cajori, dear in memory to so many of us, told me when he was revising his history of physics in 1927 that he came face to face with a profound change, that when he wrote the first edition the atom was a hard particle, twenty-five years later it was a solar system. But, my friends, what is the use of bringing a history up to date? In the past five or six years the picture has all changed and the atom is a packet or bunch of waves.

Bohr's 1913 model of one or more electrons moving about a central nucleus, like a planet about the sun, served very well to help scientists and non-scientists visualize a working model of what the atom might conceivably be and how it might conceivably work in a way to explain the phenomena of light, electricity, magnetism, etc. For example, the hypothesis was that the electron might make jumps from one orbit to another, and whenever this occurred a bit of energy was thought of as leaving the atom, a so-called quantum of energy, and having its effect as light or other radiation, or, under suitable conditions, of making its record as a spectrum on the photographic plate. But Bohr's model did not explain *all* of the known spectral lines, and this, with other failures to compass the *whole* field, caused it to be replaced by the later method of wave mechanics.

De Broglie in 1924 suggested that we associate a wave with every moving particle of matter, the easiest agreement with the ordinary or "classical" mechanics of Newton being to think of the wave as a reality, a real happening, and as occupying a certain region of space, while the particle is thought of as a material point having a definite position in the wave, let us say like the queen bee in the midst of a lot of vibrating bees.

Finally Heisenberg in 1926 introduced an entirely different procedure when he threw all models away and regarded the wave not as a physical phenomenon taking place in a region of space, but just as a symbol which mathematically resembles the expression for an actual wave, but is representative of the properties that a particle exhibits, that is, as an abstract expression in which he can read these properties when numerical values are substituted in the expression. The procedure is quite what we do in analytic geometry when we solve a geometric problem without thinking of drawing the geometric

figure, or when we find by an algebraic formula some property of a curve without troubling to draw or study the whole curve. And this method explains, I'd better say comprises or codifies, the intricate facts of radiation as has no other system. Abstruse? Yes. But understandable? Yes, at least by way of analogy with matters that we do comprehend.

It is because of such outcomes as this that Jeans says that while thirty years ago we thought that the universe was heading toward an ultimate reality of a mechanical kind, the universe now begins to look more like a great thought than like a great machine.

All this suggests to me the necessity of having such material as I have been sketching available for our teachers, probably in the form of monographs similar to the *Carus Monographs* in the collegiate and university field. The *Sixth Yearbook* makes an excellent beginning; we now need specific material of an expository nature. The National Council, alone or in affiliation with other organizations, is the proper body to put this into effect, just as you are planning to develop the admirable plan proposed by Professor Rankin, which, as I understand, is to show our teachers where this and that and the other topic or formula of elementary algebra is actually used in the applications of mathematics. Both plans are needed; both plans will be of great benefit to prospective teachers if ably wrought out.

Finally, how can a broader education of the sort I am describing be realized? Where can the time be found? My answer is, *Make the time.* It is trite to point out that just here lies the difference between cultured minds and ordinary minds, between good and indifferent teachers, but trite statements have the merit of being true. I am impressed the past few years as never before that it is necessary and that it is possible for us to *choose* what we shall do, not merely with our leisure but largely even in the line of our main activities. The question arises, first, shall *we* choose to be easy-going or to be alert; second, shall we influence our students and those teachers with whom we work in making their choices? Even after allowing for the pressure of long hours of prescribed work, we must each of us make confession that we have not made adequate use of our spare time and that in the routine of our teaching and other school duties we do not frequently enough stop and consider what is of first importance and by definite assertion of will-power discard that which is passably good for that which is better.

This is no idle dream, nor is it a vague ideal. The very existence of the National Council, the presence here and at other similar gatherings of live-minded teachers, is first-rate evidence that we have at heart ideals and accomplishments which lie outside the range of our immediate duties. I can do nothing more fitting, as I close, than to bid you persist in your good endeavor and to quote the following which came just the other day from Einstein, a scholar whom we always visualize as profoundly immersed in mysterious and unknowable mathematics:

"Concern for the man himself must always form the chief interest of technical endeavors, concern for the great unsolved problems of the organization of labor and the distribution of goods—in order that the creations of our mind should be a blessing and not a curse to mankind. . . . Never forget this in the midst of your diagrams and your equations."

News Notes

The following groups have recently become affiliated with the National Council of Teachers of Mathematics:

The Mathematics Teachers, Bay Section, California Teachers Association, Alameda, California.

The Mathematics Section, Oklahoma Education Association, Oklahoma City, Oklahoma,

The Mathematics Section, Western District, Pennsylvania State Education Association, Pittsburgh, Pennsylvania.

In addition, the Cleveland Mathematics Club at its meeting on April 21 voted to apply for affiliation.

Mathematics as a Liberator of the Spirit*

By Norman Anning, *University of Michigan*

Standing here within sight of the international boundary may I have a few minutes of your time to pay tribute to two Canadian teachers:

To John Leslie Cornwell of Stratford, Meaford, Winnipeg, who said to me, "Here is Todhunter's *Euclid;* at the back of it are 600 deductions. Work them out on that board at the back of the room and remember that I am too busy to help you find anything that you can find for yourself." As a pedagogical principle this has the approval of Descartes: "But I shall not stop to explain this in more detail, because I should deprive you of the pleasure of mastering it yourself, as well as of the advantage of training your mind by working over it, which is in my opinion the principal benefit to be derived from the study of mathematics."**

To Nathan Fellows Dupuis, at various times professor of the natural sciences, astronomy, pure mathematics, in Queen's University. He was a self-trained genius in mathematics and in the art of presenting it to students. Modern logicians would make short work of his textbooks but he had the courage to break with the Euclid tradition in a land where and at a time when such a break was heresy and he had the still-to-be-respected ability to make students work. Here is an extract from his credo:

> It appears to me that it is a prevalent custom to lay too little stress on synthetic methods as soon as plane geometry is passed, and to hurry the student too rapidly into the analytic methods. If mathematical knowledge is all that is required, this may possibly be an advantageous course; but if mental culture is, as it should be, the chief end in a university education, this customary usage is not the best one.

When President Everett asked the speaker to defend the thesis that mathematics is a liberator of the spirit, the task was accepted with pleasure because that is one of his positive convictions. The statement needs defense; there may be some present who think

* An address given at the annual meeting of the National Council of Teachers of Mathematics in Detroit, February 21, 1931.

** Smith, *Source Book in Mathematics,* p. 400.

that a mathematician is uncomfortable when he hears a student say that two points determine a line or that two lines perpendicular to the same line are parallel. And they may believe that he writhes a little until the more correct statements are forthcoming : "two points, if they are not coincident, determine a line," "two lines which lie in the same plane and are perpendicular to the same line are parallel." It ought to be admitted that long-continued effort to achieve and to enforce accurate, logical use of the mother tongue may occasionally produce in the teacher a sort of proof-reader's complex which is not entirely admirable.

If a minister were to announce the text: "Zadok the priest and Nathan the prophet anointed Solomon king, and all the people rejoiced and said 'God save the king' ", he would have by implication the following topics: God, prophet, priest, king, multitude, prayer, salvation, and he could preach until it was time to take up the collection. I am no preacher but my topic contains the word 'spirit.' I cannot define it nor prove its existence so it must be put among the axioms. Permit me the following tentative definition: Life is an adventure of spirit into matter for the purpose of collecting experience and thereby transmuting latent potentiality into consciously possessed power. Such transmutation is the business of the educator whether the object be the education of himself or of another. And mathematics must be listed among the effective agents. The very word "solve" which we use daily comes from a root which means "I set free, I strike off the shackles." Remember the lady on the prairie who said, "Heinrich, go tie the dog loose, here comes another agent."

Let me say at once that ours must be a qualified liberty. The benches in the mathematics room are hard and no cushion has been provided even for the chair behind the desk. Carved on the front of the court-house of Worcester, Massachusetts, is our motto: "Obedience to law is liberty." This matter has been aptly put by Professor Keyser in his essays on "Fate and Freedom." If I choose the sides of a triangle, I cannot also choose the radius of the inscribed circle. If I choose a cubic equation with real coefficients, I cannot choose that it shall have exactly two real roots. From the law of consequence, elsewhere called the law of Karma, we cannot escape. Congress can establish that the metre shall be 39.37 inches thereby defining the inch in terms of the standard metre but no legislation human or divine can make π equal to 3. If the beauty of the logical chain which

binds "if" to "then" has never quickened your pulses, a career in mathematics cannot bring you lasting satisfaction.

Let me sketch from composite experience what seems to me to be a most unfavorable case.

A schoolboy with shining eyes and springing step approaches his principal carrying in his hand a large diagram and four typewritten sheets.

"Hello, Mr. Stover, I've trisected the angle. You put the vertex of the angle here and the middle line of the angle—anybody can bisect an angle—along here and then. . . ."

"Hold on, hold on, this is a neat figure and doubtless what you say is correct, but I have always understood that trisection is impossible. Have you shown this to Miss Finley?"

"Yeah. She's up there now keeping in some ninth graders for their algebra. She said just what you did and showed me the place in the book but she couldn't find anything wrong with it. You construct a circle with center O and any convenient radius and. . . ."

"Wait. I cannot follow if you go ahead so fast. Let me have this for a while; I'd like to send it up to the university."

Now the farther the principal is from the classroom the nearer he is likely to be to those luncheon clubs which pump up optimism. You can therefore understand his comment as he sends the letter off: "This ought to show them that we are mentally alert down here in Toonerville."

At the college the head of the department says to a subordinate: "Here, Henry, it looks like another trisection. Better do what you can for the lad and leave a copy of your reply with my secretary."

Here there should be inserted into the story that row of asterisks which in current novels means lapse of time. Henry must answer. Depending upon a number of intangible factors his answer will be courteous or just curt. Suppose the latter.

"Principal, Toonerville H.S., Dear Sir: I am sending back your figure without looking at it. I do not need to. Trisection by Euclidean means has been proved impossible. Tell the student to read Chapter III of Dickson's *First Course in the Theory of Equations* and particularly section 32. It is there proved that he cannot trisect 120°. I need not point out to you that a method which fails for particular angles is not a solution of the problem. His papers are enclosed. Yours, etc., Henry Ashby, Prof., U. of X."

To the boy such an answer will be as welcome as frost at blossom time. In his disappointment he may say, "I should have elected Commercial; they appreciate a fellow's efforts and know how to be decently sympathetic." But if he is of the sort that may yet be made into a good student of mathematics, it is more likely that he will say, "I'll look up the reference but I *can* trisect all the angles up to 90° and I'll show them yet. Everybody laughed at the Wright brothers but they succeeded."

Can any useful lessons be drawn from an experience so cramping? Must the boy take the bitter pill or should he expect jam on it? What becomes of the gingerbread psychology which says that any self-started activity on the part of the pupil is praiseworthy and ought to be rewarded? The answer is obvious. The boy must be let down, courteously if possible, but down. He can be re-directed into profitable activity. With older people the *morbus trisecticus* tends to become chronic and, I fear, fatal.

That an unsolved problem like trisection has been good for mathematics down the ages cannot be denied. It has suggested many a worth-while problem for which we now have the solution. Without stopping for details let me mention only two: what problems can be solved with the use of the straightedge only?, what by using the compasses only? Those who wish to pursue this topic beyond the reference already given should study Hilda Hudson's *Ruler and Compasses.*

Let us turn to a more cheerful aspect of our subject.

The task of "selling" mathematics is becoming increasingly easy. There was a time when mathematics was feared. As late as the time of John Dee a mathematician was one who could cast and interpret horoscopes. There is a picture of Dee and Edmund Wright raising the devil by magic arts and among the cabalistic signs which they have drawn for their nefarious purpose are figures which are still used in the classroom for the instruction, if not for the confusion, of the beginner in geometry. That was 300 years ago. People do not now put the mathematician and the worker of black magic in the same class but it still seems fitting that mathematicians should attempt to interpret the stars; books by Jeans are among the best-sellers. The magic nowadays is wrought by means of matrices and Christoffel symbols. Within the month the goings and comings of a mild little scientist from Berlin have shared the front pages of our

newspapers with "Capone Offers Armistice," "Buckley Murderer Hanged."

The western world has scant patience with the Hindu mystic who sits in the sunshine contemplating his own stomach and meditating on the circle of existence. But it will permit the mathematician to marshal his legions of differential equations and to indulge in cosmological speculation and it will listen when he says, "Today I have arrived at the following conclusion." The discovery of the new planet is fresh in our memory. By a combination of the patience of the astronomer with the skill of the mathematician (Every Kepler must have his Tycho) the bounds of our solar system have been enormously extended and the initials of Percival Lowell have been written on the sky along with the symbols of Neptune, Ouranos, and the older gods.

The story of the planets indicated by these well-known symbols need not be repeated here. The triumph of mathematics and the international co-operation which resulted in the discovery of Neptune have been often told. This symbol does not stand for football and goalposts but is a lantern hung from the crossbar of the letter H—Herschel. This next one, Saturn, was the outermost one which could be seen by the Babylonian astronomer upon his ziggurat or, indeed, by the shepherd watching his flocks by night.

Of other ways in which mathematics has tended to set the spirit free let me mention only two. The theory of matrices, which in the time of Cayley and Sylvester could have supported the old joke: 'Gentlemen, this perfectly beautiful, perfectly useless theory . . .,' is now a boon to the student of the newer physics in which waving particles and parting wavicles gyrate in a dance which I am powerless to explain to you. The positive integers from 1 to 92, or perhaps as far as 136, have new meaning and dignity since the atom has come to be better known.

A few minutes ago I referred somewhat playfully to a quotation from a document of ancient Hebrew history. Let me in closing bring seriously to your attention another quotation from the same collection. "He made known His ways unto Moses, and His acts

unto the children of Israel." You as teachers are with flint and steel and the expenditure of vital energy sending spark after spark. When, as the days pass, there comes back to you no answering glow it is only human that you in your discouragement should occasionally say, "They are all wet and even I may be a trifle damp." Strengthen your courage with the words just quoted. There must be in every group those typified by the children of Israel who can be led by a diligent teacher to understand more or less clearly the *acts* of the Lord. What they understand they applaud and use. They make good law-abiding citizens and constitute the backbone of any community.

But once in a while you will have one of whom Moses is the type. Like the bush which was on fire but not consumed he will begin to shine by his own light, and to move forward under his own steam. Such incandescent souls, even in a democracy which writes "equality of opportunity" on its banners, cannot be prevented from becoming leaders; to leaders the Lord reveals His *ways*. One leader in your class will materially assist you in bringing the whole group out of bondage and up to the edge of liberty. The clinching argument for mathematics as a liberator is the fact that it has been done. There are teachers present who cherish the memory of such students as a part of their permanent reward and can advance St. John's argument: "These eyes have seen and these hands have handled."

Of course, mathematics is not the only avenue. Carlyle says "All roads, even this Duckpuddle (Entepfuhl) road, lead to the ends of the world."

The Thirteenth Annual Meeting

The thirteenth annual meeting of The National Council of Teachers of Mathematics will be held at The Raleigh, Washington, D.C., Friday and Saturday, February 19 and 20, 1932. The present time is not any too early to write for hotel reservations. The National Council will probably have the largest meeting in its history, and also, there is to be celebrated next year the two hundredth anniversary of the birth of George Washington.

The National Council of Teachers of Mathematics in 1930-31

THERE ARE SOME things concerning the National Council that are especially obvious and that a great many people know about, such as the annual meeting and THE MATHEMATICS TEACHER. There are some other types of the Council's activities that also deserve mention and about which I think you will be glad to hear.

The Council began a dozen years ago with a meeting, and the annual meeting is still a very important, and to many people a very pleasant event, but this program is no longer the only activity of the year. The annual meeting shares now, in its influence, with several other phases of the Council's spirited participation in the adventure of teaching mathematics. As a matter of fact, could one's vision encompass what is actually happening, he would be able to see the Council in action just about every minute—I started to say "of the working day," but that is scarcely an appropriate phrase, since the Council now extends so far geographically that every hour is a working hour.

Before long I hope that you will have the privilege of reading in THE MATHEMATICS TEACHER the report of the committee on geometry, of which Mr. C. M. Austin was chairman. That committee did a tremendous amount of work: first, in organizing its problem; and then in carrying on the investigation. The report affords the most extensive array of information that is available regarding the status of geometry in the schools of the United States, and the prevailing practices and opinions regarding the relation of plane and solid geometry. This report supplemented the work of another excellent committee, under the chairmanship of Professor Dunham Jackson, which has been working in conjunction with the Mathematical Association of America.

Among the new ventures that have been launched for this coming year is the preparation of a handbook of mathematics. This book is to include a wealth of illustrative mathematical materials and applications to which teachers may easily turn as a source book of information with which to enliven the work of the class room. Professor W. W. Rankin of Duke University is to be chairman of this committee.

Still another new departure consists of the initiation of a standing committee on cooperation with official examining boards. We who live in the Middle and Western States may at first be inclined to consider the problem of examination boards as being more local than it really is. But the policies of New York State, with one-tenth of the total population of the United States within its boundaries, affects profoundly the rest of the country. Probably there are very few teachers of secondary subjects anywhere who do not directly experience the effects of practices which may be confined in their merely legal aspects to distant parts of the country.

The *Sixth Year Book* is another notable contribution, containing, as it does, ten significant articles by especially well-known authors on "Mathematics in Modern Life." Your own library and the library of your school will be enriched by such a book.

This may be the best opportunity that I shall ever have to call attention to another, outstanding, though different, benefit which the National Council of Teachers of Mathematics enjoys. I refer to the able enthusiastic, and friendly editor of THE MATHEMATICS TEACHER, Professor W. D. Reeve. Some know him well, but there are other thousands of readers of this magazine, who enjoy its pages without, perhaps, realizing the extent to which, in a scholarly manner, the editor radiates good sense, and energy, and the inspiration and example for better teaching of mathematics. (I know full well that this paragraph would never pass the censorship of the editor's blue pencil, were it not for the fact that Professor Reeve is in China right now.)

Of a good many other members I should like to write also, and the list would be a long one of those who contribute loyally to the ideals for which the National Council of Teachers of Mathematics stands. In this and subsequent numbers you will read some of the addresses given in Detroit last February. Plans for the program to be given next year at Washington, D.C., are already well along. It requires the effort of a good many people, some working individually, some cooperatively, to maintain the level of the Council's contribution to teaching, and we are all fortunate in being able to associate with such a movement.

I wish you all the pleasure of watching the National Council grow. I wish you also the greater pleasure of helping to make the National Council grow.

JOHN P. EVERETT, *President*

Algebra and the Development of Reason*

By Susanne K. Langer, *Radcliffe College*

What I wish to say to you today is a general reflection on the subject of high-school algebra. As a layman both in mathematics and in secondary school teaching, I can speak only from two lay points of view—that of the pupil, which meets you at one end of your activity, and that of the philosopher, which you encounter at the other. Consequently I shall begin by talking about the futility and barrenness of algebra, and end, I hope, by reviewing with you its importance, interest and charm. For it is a peculiarity of the subject that an uninitiate mind can usually see nothing in it but a dry, lifeless discipline, whereas the adept sees in it the apotheosis of human reason. The mean from the former outlook to the latter is your ministry.

All sorts of systematic and skillful method have been devised for teaching young people algebra, and the amount of technique that can be imparted in a single year to perfectly immature minds is a constant source of wonder to me. But another and less agreeable mystery lies in the fact that so much algebra can impinge upon the immature mind without helping in the least to mature it. The subject has, apparently, no humanistic value—"humanistic" meaning beneficial to the development of mental power. Children who have learned algebra are not better thinkers than those who have not learned it. They have acquired a new technique, a technique of something strange and abstract; but neither they nor we can tell just why they have learned it. Its practical value is very limited: engineers, surveyors and astronomers use it. Perhaps two or three per cent of our students will be engineers, surveyors, astronomers, or members of some other profession that actually requires the technique of algebra; why must the other ninety-odd per cent of every class be drilled in this esoteric art? It is useless, formal, and difficult; at best it has the virtues of chess—it is a sophisticated game, known to sharpen the wits. We do not require a satisfactory chess-record for graduation. Why did we ever introduce the complicated game of algebra—less useful than arithmetic, which it otherwise closely resembles, less lucid than geometry, to which it is somewhat mysteriously related?

Algebra was originally included in the school curriculum for a

* An address given at the annual meeting of the National Council of Teachers of Mathematics, February 21, 1931.

thoroughly humanistic purpose: it is the only elementary discipline which furnishes material for abstract thinking, and caters to the development and pleasure of the faculty called pure reason. In arithmetic, the elementary operations can be visualized as patterns arising from the combination of various groups—we can visualize $4+5$ as two related groups of four and five objects respectively, or 4×5 as four examples of the latter grouped together—so that we may literally *see* the results of arithmetic at least in the early stages of learning; sensuous intuition feels fairly at home with number-work. But in algebra we cannot correlate our quantities with rhythmic or configured patterns; a and b have not the definite individuality of 3 and 4. Their relations are not open to inspection by the mind's eye, naïvely and instantaneously; they must be understood through the *discursive reason*. It is for the sake of this kind of reasoning that algebra was originally taught in our schools; and this, in my opinion, is its only excuse for being in the curriculum. All its other purposes—its relation to special sciences, its traditional dignity, and so forth—seem to me far-fetched in relation to the public school, or else trivial. As a foundation for a few specific sciences it certainly is not needed by literary-minded girls and boys, by future business men, lawyers, librarians, Latin teachers, kindergartners, or any number of other professions that require at least a high-school education. As a mere demand of the colleges it is silly. But as a gymnastic in pure unaided reasoning it is not only important, but unique. In that capacity it concerns not only the scientist who will operate with its formulæ, but every mind that is initiated by its subtleties into a new realm, the realm of abstract thought. For algebra, rightly understood, gives us much more than a technique for computing certain scientific problems; it increases our power of conception to an astounding degree, and liberates us from the concrete, visual and tactual habits of thought that bind the naïve mind so closely to Mother Earth.

The intellectual life of man moves between the two extremes of *concrete experience* and *abstract thought*. Near the lower limit lie those intelligent functions which we share, presumably, with the higher animals—our awareness of objects, sensuous memory, expectation, the recognition of simple means to ends. Near the upper limit we may place such intellectual constructions as pure mathematics and symbolic logic, which are entirely formal. The mental life of a

generally educated individual ought to run on a plane somewhat above the mean between these two types of thinking. He should be capable of all kinds of ideation. A cultured mind is not one that is stuffed with information, but one that is supple, adaptable to various modes of thought, trained both in sensuous discrimination and in discursive reasoning; a mind that is not bound by any one set of habits, but can shift gears according to its load. Our usual, practical thinking concerns specific situations, concrete experiences, and is weighted with a vast amount of detail; but abstract thinking is unhampered by any irrelevant detail, it is highly selective and concerns only a few essential aspects of any given experience. It can apply to any number of cases just because it does not attempt to exhaust any one of them. Therefore it is vast in its scope and swift in its passage from premise to consequence. It strips its subject-matter of all "asides" and individual peculiarities, and thus proceeds from its few premises to their conclusions with a speed that is impossible to a more elaborate process of visualizing concrete situations, which has to stop for all the details of an actual case in order to reason to another actual case. The uneducated mind is geared very low; its progress is ponderous. It pulls hard, without acquiring any momentum. It is a commercial vehicle, which does not move except for some practical purpose; it never runs for pleasure. That is because it always carries its whole cargo of sense-experience, all its known *cases* for contemplation. It does not bring them into the elegant small compass of a *generalization*. Mathematical thought, on the other hand, is concerned entirely with general propositions; its truths are equally valid for any sets of things to which you wish to apply them. It is free from all special conditions, for it is a science of things in general. Carlyle, in *Sartor Resartus*, gives one of his characters the humorous title of "Professor of Things in General." He probably was not aware of the implication that the learned Teufelsdrökh was Professor of Mathematics.

The first generalization we learn in school is that from three apples, three pencils, three boys, three girls, etc., to the number three. Most children have no difficulty with this amount of formalization; the arithmetical level of thought seems to be the natural level of civilized man. We learn to count and figure as part of our normal mental development. On this plane lies practically all of our thinking—our calculation of money, days, years, measurable goods, physical masses, frequencies and stresses. Arithmetic is the generalization of concrete

facts. A number stands for any group with a given membership. "Three" stands for all triadic collections, "two" for all couples (some mathematicians, as for instance Bertrand Russell, define the number two as "the class of all couples"). Instances of numbers and arithmetical relations may be sought in the world of concrete experience.

The educational value of algebra lies in the fact that it is a definite new step toward abstract conception. Algebraic thinking lies just above the arithmetical level. For *algebra is the generalization of arithmetic*. Its entities stand for numbers. Just as 3 means "any triad," so *a* means "any number." It does *not* mean some particular number. What is true of *a*, in the absence of further specifications, is true of every number. Thus algebra is a generalization of a higher order than arithmetic: its terms are variables whose values are not groups of things, as the values of numbers are, but whose values are numbers. You can readily see that it contributes a very special exercise to the development of the mind. This, as I have pointed out before, is its sole significance in the secondary school curriculum, the only purpose for which it should be taught to the thousands of young people who are never going to use its formulæ in solving problems. It must give them the power of abstract conception, of generalization beyond the practical limits of images and other concrete illustrations. One cannot picture the exact meaning of *a* as one can picture the meaning of 3 by a collection of just three things; for we can count three by any set of counters, but we cannot count *a*. We must know it purely by definition, i. e., we must *conceive* the meaning of *a* or *b* because it cannot be *perceived*. No other school subject is designed to expand our mental powers in this direction; the whole task of training the discursive reason, which lifts man above even the cleverest of other creatures, devolves upon the teacher of algebra.

A person who has never taught high-school algebra may be unable to estimate the difficulties of the undertaking; but a teacher of philosophy in college is in a particularly good position to estimate the results—the results of school algebra in terms of increased mental power, such as I have just described. And it is my sorry opinion that those results do not justify the years of labor in school. Philosophy does not require the use of algebraic formulæ, but it does demand a certain ability to reason *in abstracto*, to conceive of things in general, to appreciate formal relations. This ability

seems to be no more developed in high-school graduates than in uneducated children. Yet they have all passed at least one year's work in algebra, and most of them have successfully accomplished two. What have they been taught?

Why, they have been taught a great deal of algebra! Most of them know ten times as much as they will ever use. But they have not been taught very much *about* algebra, which is the knowledge they could and would use for humanistic ends. If they have ever heard of generalization, they have long since forgotten it again; most textbooks mention it on page 1 and never refer to it after that. The course hurries on to rules of procedure—rules for adding and subtracting, removing parentheses, squaring, solving for x and y—innumerable rules of the game. The children become as proficient at algebra as their mothers at bridge. As the strange occupation grows more and more familiar, they cease to wonder at the use of letters for numbers, or to puzzle their young heads about the fact that two negatives multiplied together yield a positive product. It is a peculiar fact that a proposition which at first glance looks weird or even nonsensical will seem quite reasonable once it is very familiar. The famous (or infamous) proposition that the earth is round seemed preposterous when it was new, even to many people to whom it had been duly demonstrated; now, because we are used to it, its weirdness has disappeared and we can believe it without great intellectual effort and even without demonstration. In the same way the student's original curiosity about the logic of algebra simply dies, even without being satisfied, once he has become familiar enough with the technical practice.

The mastery of algebra—even elementary algebra—requires two intellectual functions, both of which can be developed only by constant practice. They are (1) abstraction, or the recognition of general properties and relationships, and (2) manipulation, or deductive reasoning. The tendency of our educational methods is to develop the latter exclusively. This is the more unfortunate in so far as there are various other subjects in the curriculum which train the mind in the art of deduction (for instance arithmetic, geometry, physics), but none that initiates it into the difficulties of abstraction. The result of this one-sided interest is that a student who has learned algebra has acquired a new trick, but not a new insight—the content of his mind has been increased, but not its powers.

What is the reason for the cavalier treatment usually accorded to those vital problems in algebra, namely its meaning and its derivation from arithmetic? The reason for our attitude lies deep in our educational scheme, in what I believe to be a wrong academic ideal: the ideal, or perhaps the fetish, of successful performance. Not only in algebra, but in history and literature as well, our educational methods do not aim at more understanding, but at certain desired responses. Such correct responses can often be more readily procured by sheer habit-training than by any appeal to the conceptual powers. If we teach the student to operate with letters in place of numbers, and can once push him over his natural bewilderment at being asked to multiply the last letter of the alphabet by the first, he can learn, like a Robot, to operate with these meaningless symbols as though he were getting sums and products. In fact, he can learn the whole mechanics of the system much faster if he is not constantly called upon to explain the sense of his procedure. Generalization is a difficult art to the naïve, picture-loving mind; whether we can perform it or not makes no difference in the speed of our technique. An efficient teacher, therefore, cannot waste much class-room time on the derivation of $a+b$ from $1+2$ and $2+3$ and $3+4$. She expounds the matter and lets it rest. For she must get through quadratics! The colleges demand it. They do not set examination questions to test the understanding, they offer only problems of technique: to find this cube root, to expand that expression, to determine how long it takes a hen and a half to lay an egg and a half. That, as I have said, expresses a tendency of our age. We are interested in examples rather than general truths, in facts rather than principles, because our ideals are pragmatic. Activity, output, performance are our measures of a man. His behavior, and not his thought, is the object of education. (It is not an accident that our generation of psychologists has invented behaviorism.) Therefore the colleges examine only how much algebra he can "do," and not how much he can conceive. But the more technical dexterity the colleges require, the less time is there for the high-school teacher to inculcate fundamental ideas of logic—notions of the general and the particular, of variable and value, relations, and the importance of equivalent forms. These are the mathematical concepts that mark the advance of algebraic thinking over the arithmetical level, on which the mind can always resort to intuition—to images or con-

crete instances. These concepts are the new acquisitions we are supposed to gain from our traffic with *a*'s and *b*'s; their apprehension shows an actual increase in mental maturity, a step toward scientific and philosophical thought. But having introduced the study of algebra for the sake of its generality and abstractness, we now hush up that aspect of it because by so doing we can teach more algebra! Certainly there is a joker in that game.

The most obvious question to be raised at this point is whether children in high school are able to conceive abstractions. My answer is, no; but they are capable of learning to do so. Most children have very little conceptual power, and it is a sad fact that as adults they are not apt to have much more. This indicates that our education does not train our ability to form a new conception. It leaves the mind as immature and impotent as it was in the untutored state. Instead of trying to stimulate the growth of abstract thought, we arrange our educational methods so as to obviate the need of it, and in this way allow the student's intellectual strength to deteriorate more and more through disuse. Now, instead of avoiding abstract ideas, we should set ourselves the avowed task of introducing them—consciously, clearly, one at a time. There is no reason why a student of average intelligence cannot understand, for instance, that whereas arithmetic expresses relations among particular numbers, such as that $5 + 4 = 9$, algebra is concerned with relations that hold among any numbers whatever, such as that $4 \times 5 = 5 \times 4$, $20 \times 4 = 4 \times 20$, $6 \times 7 = 7 \times 6$, or generally: any two numbers, call them *a* and *b*, if multiplied together, may be taken in either order; in other words, $a \times b = b \times a$. Or consider these propositions: $2 + 3 = 5$; $4 + 6 = 10$; $13 + 16 = 29$; any two numbers added together have a sum that is a number. Any two numbers, call them *a* and *b*, have a third number, let us call it *c*, as their sum; that is, $a + b = c$; or the sum of two numbers is always a number. (Later, when the meaning of positive and negative integers has been introduced, the fact that $5 - 5 = 0$ is equivalent to $5 + (-5) = 0$, or: $a + (-a) = 0$, may be adduced to explain why mathematicians say that 0 is a number.)

If we were not in too great a hurry to get to mechanical devices in algebra and "do examples," any child of high-school mentality could learn the principles of correct generalization. Long before the student is asked to translate such expressions as: $a + b + c - 2d$,

where $a = 2$, $b = 4$, $c = 5$, $d = 3$, i.e., where definite values are assigned, he should have grasped the significance of: if $a - b = 0$, then $a = b$ (which should later be resorted to again and again to justify the rule of changing signs). These and other fundamental properties of numbers should be brought fully and explicitly to his consciousness because they are simple enough for his understanding and yet perfectly general; in short, they are generalizations which he can perform. He should, moreover, be asked to express algebraically what is common to such a list of equations as this: $5 + 0 = 5$, $6 + 0 = 6$, $3 + 0 = 3$, $1020 + 0 = 1020$. All his text-book material and class-room practice aim only in the other direction—namely at finding values for given variables. Thus he is led to believe that the most important characteristic of algebra is the presence of an "unknown." Never does it occur to him that the sums and products of arithmetic are "unknowns," and that the elementary exercises: $2 + 13 = ?$ $4 \times 5 = ?$ might as well be written: $2 + 13 = x$. $4 \times 5 = x$. In each case, find x. The presence of an "unknown quantity" does not take us from arithmetic to algebra; all our arithmetical computations involve the discovery of at least one such unknown term. There is nothing algebraic about $2 + 13 = x$, or about $2 + a = 10$. Here x and a, respectively, are written for *specific numbers*, not for numbers in general.

The details of teaching algebra from an intellectual rather than a technical point of view do not concern us just now; I have merely suggested a few possibilities to show you what sort of stimulation algebra might give to the logical interests and powers of a young student. We want to teach him to recognize algebra as a generalization of arithmetic, to appreciate the relation of a type-form to an example. Most of our students cannot even tell us why exercises in mathematics are called "examples." They have probably been told, but have been allowed to forget it again. Throughout the course they should be given some purely arithmetical examples of algebraic forms. It might even be well, at the very beginning, to avoid such mixed forms as $(a + 3)$ $(b - 5)$. A beginner can hardly escape the notion that a and b here stand for specific numbers which are not revealed. If general numbers do occur in connection with specific ones, his attention should be called to the fact that $a + 3$ means "any number plus three" and $b - 5$ "any number minus five." (Bertrand

Russell, in one of his facetious moments, defined mathematics as "the science in which we never know what we are talking about.")

Having told the class , not once, but repeatedly and insistently, that a, b and c mean any *three numbers*, equal or unequal to each other, great or small, we come to the next algebraic concept, that of *relations among numbers*. The structure of the number system should here be pointed out—namely that every number has a successor, or next neighbor on the right, and every number except 0 has a predecessor, or next neighbor on the left. Thus the number series is infinite in one direction. The children will readily see that, since there is no upper limit to this series, it is true that if any two numbers be added together the number which is their sum must be somewhere in the series, so that it is quite safe to say quite generally that for every a and b there is a c such that $a + b = c$; but that the same is not true when we *subtract*, or work in the other direction, for here the outfit of available numbers is limited. There are no numbers below 0. We cannot say that any number may be subtracted from any other number and the result will always be a number; that $a - b = c$ always. Here our generalization of arithmetic seems to break down; $a - b = c$ is not generally true as long as 0 cuts off one end of the number series.

Here we arrive quite logically at the negative numbers. The thermometer, which may drop below 0, is a familiar household example of purely relative number. By means of the relative numbers we may take away and take away without specifying any limit such as 0. We simply go into debt; the result of a subtraction may take us below 0. Thus the general proposition, for every a and b there is a c such that $a - b = c$, has been saved. (I have no doubt that in using the expressions $a + b = c$ and $a - b = c$ one must train the student—perhaps with considerable pains—to realize that c is *not* a particular number, so that we are *not* asserting that the sum of two numbers is the same as their difference.)

This matter of negative numbers is well explained in most of our textbooks, but often a good deal of confusion remains in the student's mind regard the signs + and —; for + may indicate the place of a term in the number series, meaning that it is above 0, or it may be a sign of addition. Here one must tell the class explicitly that + is used for two purposes and has two distinct though related meanings:

+ *belonging to a number* puts it to the right, or as we say "above," the number 0, and + *between two terms* indicates a way of putting them together; in the old days when all numbers except 0 were above 0, this operation always took us further in the plus direction, wherefore the sign of this operation came to be used also as the above-zero sign, and the sign which took us toward 0 became the below-zero sign for relative numbers. Both signs at present have double meanings. Unless these are clearly understood, the student cannot be expected to comprehend as well as believe that $a - (-b) = a + b$; and in mathematics we should be trained to comprehend everything that we believe.

The fact that numbers stand in definite relations to each other not only makes it possible to perform operations, *i.e.*, to determine any number as so-and-so far from any other number, but it also enables us to express the same number in many different ways; to write a as the number which is so-and-so far above b, or below c, or lying evenly between any number and its negative, as 0 does. We may describe 2 as the successor of 1, i.e. as $1 + 1$, or as the predecessor of 3, i.e. as $3 - 1$, or as the number that must be added to 4 in order to reach 6, i.e. $6 - 4$. This takes us to the third great principle of reasoning, which we *use* in all mathematics, and should recognize explicitly through the generalized formulæ of algebra: *the principle of equivalent forms*. Why do all the examples at which we labor in school take the form of equations? Because the recognition of equivalent forms is the whole art of deduction. This art can be learned only by patient practice, and this the standard course is designed to give; for at this point our algebra course in school begins. But all too often the practice is performed purely as a chore, without logical insight, and the importance of the principle is not understood at all.

The principle of equivalent forms is the basis not only of mathematical deduction, but of all scientific analysis. Water = H_2O; we cannot relate water to other chemical structures until we think of it under this special form, as a compound of certain elements. Likewise we cannot measure a velocity without conceiving it as equivalent to a certain relation between mass, distance, and time. And not only in physics, but also in metaphysics, what are we really doing but seeking some form under which the various items of experience may be seen in relation to one another? Their obvious forms do not allow this; but where one way of seeing them baffles us, another may show

without difficulty how they fit into one system. Even in everyday life we are constantly resorting to various forms of the same thing; it would be very hard to buy small articles at absurd fractions of a dollar, if we could not treat the dollar under its equivalent form of one hundred cents. Ingenuity in thinking—whether in practical, scientific, or philosophical thinking—is primarily the art of recognizing widely different things as different *forms* that amount to the same thing—as ice, vapor, and water are forms of the same thing, or equivalents—or else discovering new ways in which a familiar thing may be treated so as to reveal some hitherto unknown relation; as lightning, for instance, could not be fitted into a system of physics until it was thought of as an electrical discharge. Algebra offers great possibilities for ingenuity; type-forms of equivalences should not merely be learned and applied, but whenever they are simple enough they should be derived or analyzed in class.

To present a detailed programme for the logical study of algebra is beyond my ambitions; that is a task for experienced teachers and educational theorists. I am merely pointing out a few of the fundamental ideas which mathematics, and algebra in particular, should contribute to the growth of the human mind. Of course I realize that any course based upon these requirements would have to proceed much more slowly than the standard course. There would be lessons spent on finding examples of the order of relative number—the thermometer, credits and debits, right and left, before and after (though it is a peculiarity of our chronology that, being invented and adopted before the introduction of 0, it has no year 0, and thus does not conform to the pattern of the algebraic number series. A modern calendar would certainly denote the year of the nativity as 0). There would be questions and discussions of the meaning of signs, of the reasons why $a + b$ is an expression whereas $a \times b$ is a term, what it means to multiply by a negative number, i.e., apparently, to take a term a negative number of times, and the meaning of "constant" and "variable." Some of these problems—not merely their answers, but the problems themselves—would have to be carefully circumvented at the beginning, so as not to swamp and bewilder the blundering, uninitiated minds. The course would move very slowly at first; but it would make algebra appear as a study, not a game. Examples would become examples *of* something, and formulæ would be summaries, not rules.

Whether algebra can be taught on these principles to the average

adolescent mind, I do not know; the scheme may be Utopian. Only careful pedagogical experiment can decide such a question. But if algebra cannot be imparted in this or some similar way to everybody, then it should not be a required subject in a general arts course. In the technical schools, where ability in computation is the chief aim of algebra, the present method of presentation is probably by far the best; but in the classical course it is out of place. There algebra should be taught from the logical standpoint to those who can understand it, and not at all to those who cannot. It should be elective, like Greek. I do not mean by this that it should be recommended only to children of marked mathematical talent; on the contrary, all children with a superior intelligence quotient should be urged to study it, the practical, concrete-minded ones as well as those who are naturally abstract thinkers, for a lack of natural bent is no reason for neglecting a faculty which normally requires some education. We do not advise awkward children to avoid dancing and gymnastics, but seek rather to help them over their ineptitudes. Any mind which is generally active and keen should be developed as far as possible in several directions. The child whose thinking is preponderantly verbal should have generous training in sensuous discrimination and visual judgment so as not to let his lesser talents deteriorate; there is, of course, no sense in making him use visual methods where his own are natural and suitable—that would be like reforming left-handedness, an interference with nature—but he should be given plenty of tasks that absolutely require visual attention. Likewise a child whose mind is above the average, but whose thinking is predominantly picturesque and particular, should be systematically taught the art of generalization, lest his logical powers be lost completely through the unconscious habit of evading logical tasks.

The average individual does not take kindly to abstract thought; this weakness has been recognized by our contemporary writers and educators, and in their anxiety to impart huge doses of information to thousands of average people they have worked out a remarkable technique of avoiding abstractions. Thus it has become a fashion and almost a mania to present difficult logical matters—relativity, quantum-theory, Gestalt-theory or what-not—in all sorts of enticing imagery. This popular practice has the pernicious consequence of leading people to suppose that they understand anything which they can associate with an interesting mental picture or of which they have

been shown an example. Naturally, if they believe that the thoughts of Einstein can be put at their command through simple language and concrete illustration, they will regard difficult mathematical abstractions as gratuitous and silly. Thus our own constant catering to the preferences of the untutored mind has created a sort of horror and suspicion of abstractions which stands in the way of many a student's natural interest and pleasure in algebra or logic. If mathematics is ever to be a real contribution to the intellectual development of your pupils, you must combat this attitude by your own attitude—by emphasizing the intrinsic interest of a general truth quite as much as the usefulness of certain computations, by recognizing the intellectual virtues in algebra—its superiority to arithmetic in point of generality, its perfect logic, its relation to philosophy. If we teach algebra as part of a humanistic course, it must be because we hold an academic ideal of easy, correct, and unrestricted thinking. A trained mind likes to think. An unskilled mind is afraid of the chore. If the high school is to have any deep effect on the lives of its students, it must contribute not only to their stock of subject-matter for thinking, but also to their powers of thought. Of all school disciplines, algebra is the special training ground of reason. It offers the first logical technique, requires the first sophistication of reason, and is the antechamber of science and philosophy.

News Notes

Upwards of 300 teachers including seventy-five members of the Council were present at the meeting of the Mathematics Section of the Oklahoma Education Association on February 6, 1931, the chairman of the section being Martha Liddell of Bartlesville, the secretary, Virginia Parks of Tulsa. The program was as follows:

New technique and new content in algebra. W. W. Hart, University of Wisconsin.

Geometry in the English Schools. Gertrude Tesh, Tulsa (exchange teacher from London).

Answers to questions about teaching the adopted text in geometry, by the author. J. O. Hassler, Norman.

A plan of introduction to algebra. Aggie Boyet, Wilburton.

Report of the Committee on Geometry

EARLY in 1929 a committee was appointed jointly by the Mathematical Association of America and the National Council of Teachers of Mathematics, to study the feasibility of a proposal that college entrance requirements in geometry should be modified so as to bring about the more extensive introduction of courses including the essentials of plane and solid geometry in a single year's work, in place of the traditional year of plane geometry. The Committee begs leave to report as follows:

There is a wide-spread conviction held by a group of teachers impressive both as to numbers and as to scientific and educational standing, that the proposed rearrangement will be a fundamental improvement in the teaching of geometry. The leaders of this group are teachers who believe in the importance of mathematics as part of a liberal education. Their desire is not to diminish the amount of time given to the teaching of geometry, but to increase the effectiveness of that teaching. Their chief concern is to remedy the situation created by the progressive disappearance of solid geometry from the course of study of the average high-school pupil, and to bring some important parts of it into the program of pupils who under present conditions would learn nothing of it at all.

The Committee believes that they should be given all practicable support and encouragement in their endeavor to solve the problems involved in the rearrangement, and believes in particular that participation by the College Entrance Examination Board will be a peculiarly effective and highly essential influence toward the accomplishment of the purpose.

At the same time it is clear that very many teachers, probably a great majority numerically, are unconvinced as to the desirability of the change, and that anything like a complete replacement of the existing treatment of plane geometry is not to be anticipated until the new courses have had ample time to establish and justify themselves.

There are conspicuous instances already of activity along the lines of the proposed revision. Carefully planned one-year courses in plane and solid geometry have been adopted as standard in the public high schools of Denver, and as alternative to the regular course in plane geometry in the schools of Oakland. (It may not be inappropriate to point out that although somewhat overshadowed by a larger neighbor, Oakland itself is a city of approximately the same size as Denver or Providence.) Definite steps toward the introduction of similar courses have been taken by the organization representing the

teachers of mathematics in the State of Kansas, by the schools of Detroit and New York City, and by the Secondary School Board, an organization of influential private preparatory schools in the East.

As to the attitude of the colleges, letters of inquiry were addressed by the Committee to representatives of twenty-four institutions which make extensive use of the examinations of the College Entrance Examination Board, and rather more than half of the replies indicated favorable interest in the project either on the part of the departments of mathematics concerned or on the part of some of their individual members. The present discussion, as far as it relates specifically to college entrance requirements, appears to be an outgrowth of communications published in the *American Mathematical Monthly* in 1928 by Professor Beatley of Harvard University and Professor H. W. Tyler of the Massachusetts Institute of Technology.

The launching of the Committee itself was a result of the active personal interest of both President Young of the Association and President Barber of the Council.

The present standard course in geometry is still powerfully influenced by the tradition of Euclid, who wrote before the methods of algebra, trigonometry, analytic geometry, and the calculus were available. It is inevitable therefore that the treatment of the course should be subject to perpetual re-examination, in the light of modern methods for the analysis of its mathematical content, as well as of prevailing conditions in the body of students to be taught. It is noteworthy that the teaching of geometry was the dominant theme of the recent meeting of the Council in Atlantic City, as well as of its *Fifth Yearbook*. The fact that the consolidation of plane and solid geometry is only one of a great variety of changes that might be proposed, is all the more reason for concentrating effort toward the solution of this particular problem while the attention of teachers is focussed upon it.

On the other hand, if geometry in the schools has more than merely utilitarian value—and the sponsors of the present movement are among those who believe that it has—that value is bound up with some degree of continuity in its teaching. Much of its importance lies in the fact that it is a part of the common background and experience of educated men. The Pythagorean theorem is significant not merely because it is true, but because intelligent people everywhere know that it is true. Changes in method and content must

not be so radical as to make one man's geometry unrecognizably different from another's. The skill acquired by experienced teachers in years of service is not to be redirected too abruptly. Those colleges whose avowed purpose is to attract the best pupils from small and remote schools must continue to recognize the traditional methods which will prevail in those schools for a long time to come.

Although it is not the function of this committee to formulate detailed recommendations, it may be appropriate to set down some further considerations which bear on the working out of the problem.

A most essential feature of the Board's examinations is the presence of really substantial "originals." The power exhibited by the better candidates in attacking such originals bears eloquent testimony to the ability of the candidates and to the effectiveness of the instruction that they have received. Unless the revision can be carried through without detriment to this feature of the requirement, its purpose will have been defeated at the outset. If it were demanded that the range of the course should be largely increased without diminution of its average depth, the attempt might be dismissed as obviously hopeless. But if it is granted that the maximum depth rather than the average depth is the important thing, the task may not be impossible. Specifically, it may be recalled that originals to try the powers of any candidate can be based on the content of the first three books of plane geometry.

If the pupil can once be brought to appreciate what consecutive deductive reasoning means, he will have a healthier attitude toward geometry if he then learns more facts than are completely proved, with frank and explicit recognition of logical omissions, than if he tries to believe that the last word has been said on the subject when the last page of the book is reached. To admit that geometry as we actually know it, personally and individually, is far from abstract finality, is not to surrender its essential character, but to bring it closer to the rest of human experience.

The detailed choice of material will naturally be a subject for extensive study and debate. There will be differences of opinion as to the amount of stress to be placed on the content of the first book of solid geometry, according as it is considered to be the most important or the most repellent part of the subject. A distinction may perhaps be made between the numerous and elusive proofs whose precise form is significant only in connection with a particular order of propositions,

and the few characteristic and substantial demonstrations which erect the first three-dimensional structures from two-dimensional elements. The measurements of solid figures will undoubtedly be included, but is susceptible of treatment varying all the way from theoretical prominence to incidental computation. The geometry of the surface of the sphere may be regarded as of prime importance in a science which still goes by the name of earth-measurement, or as comparatively inaccessible. A no less fundamental problem will be the selection of the parts of plane geometry to be omitted.

The amount of ground that can be covered is not to be estimated by a mere count of propositions. Much time and maturity of experience are required for the assimilation of fundamental concepts, which when once mastered appear so obvious that it is hard to make due allowance for the process of their introduction. Much may be accomplished by informal and repeated reference to the concepts of solid geometry early in the course, so that they are long since familiar when they first appear in the formal order of propositions. Or it may be that intuitive geometry in the junior high school is an essential foundation for the satisfactory carrying out of the whole program.

It is not to be overlooked, on the other hand, that an integral part of the project is the formulation of a second course for the senior high school, to be articulated with the first and to include the parts of plane and solid geometry not covered by the first year's work. While not a matter of concern to the numerical majority who take only one year of demonstrative geometry, adequate provision for the scientifically and professionally important minority who go further is a vitally essential part of the undertaking.

Various suggestions have been made for utilizing this opportunity of bringing about a closer co-ordination between geometry and other branches of mathematics. It has been urged, for example, that more extensive use should be made of algebraic methods. On the other hand, it may be worth while to point out that the "paradox" of extraneous roots in algebra is merely a matter of failing to distinguish between a proposition and its converse. It has been proposed that numerical trigonometry should be included in the geometry course as well as in the course in algebra. The work in geometry may both profit by the use of co-ordinates and clarify the pupil's understanding of them if he encounters them elsewhere. Without any attempt at

a premature introduction of methods of the calculus, some hints as to the nature and efficacy of those methods may be left where the intelligent student will find them.

The working out of these suggestions is largely a matter for teachers and textbook writers rather than for the framers of formal entrance requirements. It may be possible nevertheless for the latter also to exert some influence toward putting them into effect, without impairing the distinctive character of the course in geometry as such.

A necessary auxiliary to the satisfactory establishment of the new courses will be the preparation of suitable textbooks. It is to be anticipated however that such texbooks will be forthcoming, if it becomes clear that the attitude of schools and colleges is such as to render them commercially feasible.

As a means of putting into effect the proposals under discussion, the present committee suggests that a small committee, of three or five members, be appointed by the College Entrance Examination Board, to formulate appropriate requirements for tentative adoption as alternatives to the present Mathematics C and Mathematics D. A small committee is recommended because the speedy and vigorous adoption of a temporary working basis is regarded as more important than an attempt to reconcile all varieties of opinion in the present transitional stage. The committee would naturally be authorized and expected to avail itself of the advice of persons outside its membership. It is further suggested that the same committee be asked to construct examination papers under the new requirements for a period of years, subject to suitable revision and control, instead of calling upon the regular examiners to assume this additional responsibility, and that it be regarded accordingly as a standing committee to observe the progress of the experiment and to make subsequent recommendations as occasion demands.

Respectfully submitted,

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Mathematical Powers*

By W. W. Rankin, *Duke University*

A Complete answer as to what constitutes mathematical ability awaits some unborn triadic genius, who is at once philosopher, psychologist, and mathematician. The mathematicians say, *we think;* the psychologist inquires, *how do you think?* the philosopher asks, *what do you think?* And still we await the triadic genius. In the meantime if our classification is mathematician, remember, we think; and furthermore we must keep it up while waiting for the unborn genius.

Before we can hope to find mental powers which we can call mathematical, it will be necessary to discuss the nature of mathematics. Many definitions of mathematics have been attempted, many more statements about mathematics and its nature have been made. In a splendid collection of definitions and statements about mathematics by Moritz in *Memorabilia Mathematica* there are 1173 such quotations from the pens of well-known scientists and mathematicians. All seem to agree that mathematics has something to do with thinking. Keyser remarks, "Mathematics is the sole domain of certainty." Sylvester said, "The object of pure mathematics is that of unfolding the laws of human intelligence." De Morgan said, "The moving power of mathematical invention is not reasoning, but imagination." Bertrand Russell makes this statement, "Pure mathematics was discovered by Boole in a work which he called 'The Laws of Thought.' His work was concerned with formal logic and this is the same thing as mathematics." A favorite definition of mathematics is that given by Benjamin Peirce, "Mathematics is the science which draws necessary conclusions." The most recent definition is just from the pen of Professor Keyser, "Regarded as an enterprise mathematics is characterized by its aim, which is that of establishing hypothetical propositions."

It will not be amiss in passing to give Professor Keyser's proposal for a definition of science: "Science regarded as an enterprise is characterized by its aim, which is that of establishing categorical propositions."

* Read before the National Council of Teachers of Mathematics, Detroit, Michigan, February 21, 1931.

If the thousands of definitions and statements about mathematics should be arranged in chronological order, the careful reader would observe an enriched conception of the nature and significance of mathematics as the centuries have passed. Mathematics may properly be thought of as having begun when the ancient Greeks set up the idea *If . . ., Then . . .*, in the 6th century B.C. although considerable progress had been made in the development of number systems and in informal geometry elsewhere at a much earlier date.

Before proceeding further I should like to discuss briefly a very erroneous conception of mathematics which seems to be prevalent in the minds of many, and even in the minds of some who teach mathematics. It is against the erroneous notion that mathematics is a tool that I wish to raise my voice in vigorous protest. Now a tool is something which may be picked up and laid down at will. We do not think of the hand or the eye as a tool, but rather are they parts of our very physical being. Let us take the number system, one of the simplest features of mathematics, shall we regard the number system as a tool? Yes, there are some who so regard this intellectual achievement of man. It is second to no other achievement; it required four thousand years to perfect the number system. Once a person has learned the number system, it becomes an integral part of his intellectual being. He cannot pick it up and lay it down, he is permanently throughout his life a different person because he has the consciousness of being able to count and use the number system with confidence and freedom.

The mere fact that the mathematicians have a highly developed symoblism and are able to explain their conclusions in formulas must not be taken to mean that the formula is a tool. A formula is a conclusion, it is dynamic, it is the life-giving principle of the ideas involved. The symbols within the formula may properly be regarded as tools. It is just here that many have fallen into the error of thinking of the formula itself as a tool.

It is far more important that we should correct this misconception, that mathematics is a tool, than would appear. First, because mathematics is unalterably opposed to error, and second, the mathematician is permanently committed to set things in their proper relations. This erroneous notion has already greatly discredited the value of mathematics in the educational system. There are many educators who have to do with building up the curriculum of the schools and

colleges who regard mathematics as a tool. No study can rightfully be considered a tool, and least of all can mathematics be thought of in this manner.

Mathematics has made the sciences and engineering prosperous, and if we can accept the definition above, "mathematics is the science which draws necessary conclusions," it has greatly enriched man's intellectual life where progress is dependent on the power to draw valid conclusions. It is an ironical fact that among the scientists and engineers, the chief beneficiaries of mathematics, are to be found those who would discredit her. Mark Twain once humorously remarked, "The difference between a dog and a man is, if you pick up a poor dog and make him prosperous he will not bite you."

There is also grave doubt in my mind whether we should continue to use the expressions, *Pure mathematics* and *Applied mathematics*. I am confident that they create wrong impressions, and undesirable notions about mathematics. For the sake of definiteness and for the purpose of discussion, let us hold in mind the definition of mathematics given by Peirce, "Mathematics is the science which draws necessary conclusions." What are some of the mental traits which may be characterized as indicative of mathematical ability or symptoms of mathematical powers? I should like to submit for our consideration the following six: (the order does not indicate the relative importance in the mind of the speaker, nor is it the purpose of the speaker to become psychologist and tell to what extent these traits may be inherited, or to what extent they may be acquired, or how far they may be developed.)

First let us consider the *power of observation*. The power to fix one's attention on one or more objects or ideas and to recognize the likenesses, or differences, or peculiarities is a fruitful source of finding facts. A laboratory is an observatory. Intelligent observation leads to recognition of order, methodical arrangement, and harmony, or to the absence of these. The empirical formulas grow out of observation. Recognition of arithmetic, geometric, and algebraic forms is essential to success in mathematics. The student who fails to observe the forms of algebraic equations, and just how they are bound together is doomed to failure.

The highly developed symbolism of mathematics makes the observation of form of paramount importance. Endless manipulation will do little towards recognition of form. Suppose as we write down the

quadratic equation $ax^2 + bx + c = 0$ and as we solve it saying, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ we should observe that this single little formula includes all the real rational numbers, all the imaginary numbers, all the complex numbers, and most of the useful irrational numbers. Or again let us suppose the observation is made that the binomial theorem which expands $(a \pm b)^n$ is the same form found in the formula for computing compound interest, $A = P\ (1+r)^t$ a fundamental law in modern business. These forms become dynamic, and life giving principles in the thought world, and in the daily workshop. Geometric forms meet our eyes in whatever direction we turn, and yet our abiding faith in the textbook in geometry results in failure in the development of the powers to observe definitely the attractive combinations of forms wrought into the works of the artists and the architects. The Greeks excelled in the study of geometric forms.

The trained observer often discovers some new relation while looking for some particular relation. It is said that Sturm while observing the motion of a compound pendulum came upon the idea which led to the well-known theorem concerning the roots of equations and which bears his name. Countless other illustrations might be cited. A. N. Whitehead says, "To see what is general in what is particular, and what is permanent in what is transitory is the aim of scientific thought."

Next let us consider the *power to balance values,* or the *power to set up the law of equivalence.* The idea of substitution plays an important part in almost every phase of life. In the realm of thinking it extends and strengthens the power of drawing conclusions. Alertness in discerning just when and under what circumstances there is a relation of equivalence is unmistakable evidence of mathematical powers. That mathematics deals with the power to ascertain the relationship of equivalence in a highly technical way does not mean that this power is different when applied in the study of mathematics. In mathematical thinking the equation typifies the law of equivalence and illustrates the law in its perfection.

The best scientific thought tends toward the setting up of the law of equivalence. The research students in the problems of industry, commerce, and politics are slowly but surely evaluating those forces which control the industrial, economic, and political life of a nation.

In so far as they are able to discover permanent values and relationships, just to that extent they are able to set up the universal law of equivalence. Every road in life seems to lead to innumerable possibilities, but through the principle of equivalence it seems that there has been set up a balance of some of the probabilities. It is this that makes life livable and enjoyable, and holds nations together. In youth there is an abundance of evidence of this power of setting up the law of equivalence as in their rules of play, in their trading, their comparisons of size, strength, etc.

Another characteristic of mathematical power is the *power of transformation.* The principle of transformation is such an intimate part of our daily life it would be impossible to appraise it at its full value. By careful observation the scientists have been able to express in mathematical formulas many of the transformations which take place in nature, the law of gravitation, the laws of growth and decay, laws of planetary motion, etc. At times the skill of the observer has been so ingenious as to suggest magic on the part of the observer. The transformations with which the mathematicians are concerned largely deal with the laws of thought, and often are hypothetical in character. This naturally enough would suggest to many that there is magic. But least of all is the mathematician interested in magic.

The four fundamental operations may be classified as transformation principles. This includes the manipulative side of mathematics which already claims more than its share in the time allotted to mathematics. The transformations wrought by mathematicians are characterized by their definiteness, as $(a+b)^2 = a^2 + 2ab + b^2$, $\log 100 = 2$, etc. The differentiation of a function of x with respect to y marks one of the greatest achievements of the human mind.

The words youth and transformation are so intimately associated that they seem to be synonyms. The precision of the mathematical transformations often fail to attract youth which is apt to prefer capricious transformations.

Let us consider the *power of imagination.* Imagination is the creative faculty of the mind. One of the most dominating ideas in modern thought is that of imagining that if things were set in this or that order all would be well. The imagination is the motive power in experimental work, and every untried phase of life is experimental. Reason must feed on something and it is the imagination which supplies the raw material for reason to consume. Fortunately when

reason has consumed the raw material and turned out a conclusion, the imagination suggests some new combination of this conclusion with other conclusions, and reason again brings forth a conclusion hitherto unknown. Thus it is that the tower of knowledge rises higher and higher.

The subtle power of the imagination if encouraged to have free play in mathematics is far more potent than might be supposed. I think De Morgan was right when he said, "The motive power of mathematical invention is not reason but imagination." The fact that the imagination appears to be capricious and usually is incapable of explanation has led some mathematics teachers to discredit its value.

Students can rather easily be taught to reason. A skillful teacher can force a student to a conclusion or to a point where the student refuses to think. But to explain what suggested the idea to the teacher, and bring a student to the point of imagining something relative to the discussion is a challenge of the first magnitude. It is the imagination which furnishes the hypotheses, the axioms, the postulates, and suggests methods of proof. This characteristic of the mind is the same that led the Wrights to imagine that they could fly. It is my faith that some day we shall understand this subtle power of the mind well enough to cultivate and direct it. It will mark a new era in education.

The most generally recognized characteristic of mathematical power is the *power of reason*. Some degree of mental alertness in conceiving a logical sequence will be found in every person who can lay just claim that his thinking is rational, and his conduct normal. If there is a marked degree of alertness in conceiving logical sequences we also find a type of individual who is attractive. As a father I can truthfully say that there is no pride that brings more joy to the parent than to observe evidence of logical order in the child's thinking. Second to this characteristic of logical sequence, I think most parents would place a healthy imagination. Reason has properly been made Prime Minister in mathematics. This, though, must not be taken to mean that there are not other high officials in the kingdom.

The imagination presents hypotheses, and observation furnishes facts from which reason later draws conclusions, often much later. It took reason three hundred years to check the hypotheses supplied by the imagination, and the facts furnished by observation and draw

the conclusions found in plane geometry. It took reason one hundred and fifty years to check the hypotheses proposed, and the facts observed and to place calculus on a firm basis. It required the mind of a Cauchy capable of dealing with differences of a higher order than the first, to make calculus understandable and explainable. Yet Cauchy's friends considered him very fussy about trifles. This fussiness about trifles by Cauchy proved to be of inestimable value to the mathematicians, and to all of the sciences, and civilization in general.

It required more than four thousand years for reason to set in order the imaginations and observations necessary to give us a satisfactory number system. Many other similar illustrations might be cited. According to our definition this power of reasoning or drawing conclusions is mathematical in essence. The author of our definition remarked that "Mathematics belongs to every inquiry, moral as well as physical." The imagination is the *If-ness* of mathematics, and reason is the *Then-ness* of mathematics.

The sixth and last symptom of mathematical power under discussion is that of *Distinctness of conception and perception.* If we are to work with ideas and objects intelligently there must be distinctness of perception and conception. If we are to give and follow directions there must be clarity and definiteness. It is a well-known fact that there is a close correlation between reading ability and mathematical ability.

The concept of the fraction is very difficult especially when we pass from the concrete arithmetical idea into the general idea of a fraction as found in algebra. When the student cancels the a in the numerator with the a in the denominator in such a fraction as $\frac{a+b}{a-b}$ it is evidence of a lack of definiteness of conception of the real nature of fractions.

Distinctness of conception is characterized by precision of statement and hence this symptom of mathematical power is easily detected. It is here that distinctions must be made between such ideas as definitions, axioms, postulates, undefined elements, the arbitrariness of the symbols, etc.

These six symptoms of mathematical powers: *observation, setting up the laws of equivalence, transformation, imagination, reason, distinctness of conception* and *perception,* have been discussed very briefly. They are six of the most outstanding characteristics of mathematical ability, but no claim is made that these are the only ones.

The universality of these characteristics should serve to give teachers of mathematics a consciousness of the bigness and dignity of their work. The permanency of the principles dealt with, the definiteness of the methods of proof, the need for clear thinking, the possibilities of the transformations in the thinking of youth, the fun of imagining things, offer to the teacher of mathematics a challenge of large dimensions.

John Napier

1550-1617

THE SCOTCHMAN NAPIER was known to his contemporaries for his work in theology and in mathematics. He was active in his inventions or projected inventions for use in warfare, writing of burning glasses, "devises of sayling under water," and artillery that would "clear a field of four miles in circumference of all the living creatures exceeding a foot in height." His neighbors reported that he practised black magic, and they told of a coal black rooster that was his similar spirit.

Napier's Rule of Circular Parts has been called by Cajori "perhaps the happiest example of artificial memory that is known."

His "rods" or "bones" were to facilitate multiplication by a mechanical process, while his logarithms replaced the operations of multiplication and division by the simpler ones of addition and subtraction. In his work on logarithms he seems to have drawn his first inspiration from trigonometry where he was familiar with the formula $\sin A \sin B = \frac{1}{2}(\cos \overline{A-B} - \cos \overline{A+B})$. This method of finding the product of two sines by the difference of two cosines was known to Pitiscus and Clavius under the name *prosthaphaeresis*.

For detailed accounts of Napier's work and influence, consult the *Napier Tercentenary Memorial Volume*, edited by C. G. Knott, London, 1915. For source materials, the *Source Book in Mathematics* edited by D. E. Smith offers excerpts of translations of Napier on the Table of Logarithms, Delamain and Oughtred on the Slide Rule, Napier on the Napier Rods, Pitiscus and Clavius on Prosthaphaeresis.

The Need for Being Definite with Respect to Achievement Standards*

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THERE ARE TWO purposes in this brief discussion: first, to present evidence that there is need for being definite in stating to pupils and parents what changes we are trying to make in pupils during a given period of time; and second, to suggest certain criteria to serve as a basis in securing such a list of essentials.

By way of introduction let us consider data to answer two questions: (1) How well do pupils in grades 5 to 12 learn some of the things we are trying to teach them? and (2) what mastery of fundamentals is shown in tests given to freshmen in a university? The following test, consisting of 100 simple tasks in arithmetic, and the long table accompanying it, relate to the first of these questions.

100 TASKS IN ARITHMETIC

I. *Addition*

1. 8, 7, 3, 6, 9, 5, 8

2. 463, 877, 539, 198

3. $386.85, 96.66, 6.57, .98, 100.00, 5.94, 60.00

4. $1\frac{7}{10}$, $\frac{3}{5}$

5. $\frac{1}{8}+1\frac{3}{16}=$

6. $17\frac{2}{3}$, $5\frac{1}{4}$

7. $\frac{2}{3}+\frac{1}{2}+\frac{7}{10}=$

8. $9\frac{1}{4}$, $27\frac{1}{2}$, $8\frac{9}{16}$

9. .07+5.23+8.29+1.40=

10. 7 yd. 2 ft. 9 in.
3 yd. 1 ft. 8 in.

II. *Subtraction*

11. 1124, 742

12. 880.75, 785.78

13. $\frac{11}{12}-\frac{1}{6}=$

14. $8\frac{1}{2}$, $5\frac{3}{4}$

15. $2\frac{3}{4}-\frac{2}{3}=$

16. \$5.04−18¢=

17. 9.752−6.007=

18. \$32−\$6.58=

19. $9\frac{1}{2}-2.2=$

20. 8 yd. 1 ft. 8 in.
2 yd. 2 ft. 9 in.

III. *Multiplication*

21. 95, 82

22. 609, 40

23. 769, 708

* This paper was presented at the Detroit meeting of the National Council of Teachers of Mathematics, February 20, 1931.

24. $\frac{3}{4} \times 60 =$

25. $\frac{5}{4} \times \frac{3}{7} =$

26. $\frac{3}{8} \times 1\frac{2}{10} =$

27. $45 \times \frac{2}{5} =$

28. $20\frac{3}{8} \times 12 =$

29. $1\frac{1}{3} \times 2\frac{1}{4} \times \frac{3}{4} =$

30. $\begin{array}{r} 4.928 \\ \times\ 3.2 \\ \hline \end{array}$

The answers in the following examples have not been "pointed off." Put the decimal point in each answer where it belongs.

31. $20 \times .20 = 400$

32. $1.6 \times 0.3 = 48$

33. $0.5 \times 5.0 = 250$

34. $0.245 \times 2 = 490$

35. Does 1.2×0.5 equal 6.0 or .60 or .060 or 60?

36. 3 times 2 gal. 3 qt. 1 pt. =

37. $\frac{1}{2}$ of 3 bu. 2 pk. 2 qt. =

IV. *Division*

38. $21\overline{)882}$

39. $78\overline{)12418}$

40. $85\overline{)343400}$

41. $\frac{3}{4} \div 4 =$

42. $4\frac{1}{2} \div 8 =$

43. $2\frac{1}{2} \div 4\frac{1}{2} =$

44. $3\frac{3}{4} \div \frac{3}{4} =$

45. $6\frac{1}{2} \div 2.5 =$

46. $.004\overline{).0284}$

47. $.34\overline{)105.4}$

The answers in the following examples have not been "pointed off." Place the decimal point in each answer where it belongs.

48. $.123\overline{)560.88}$ — 456

49. $1.23\overline{).56088}$ — 456

50. $12.3\overline{)56.088}$ — 456

51. $123.\overline{)560.88}$ — 456

52. Does $4786 \div 10$ equal 4.786 or 47.86 or 478.6 or 4786?

53. Divide 6 lb. 2 oz. by 5.

V. *Fractions, decimals, and per cents*

Write each of the following as per cent:

54. $\frac{7}{100} =$

55. $\frac{3}{4} =$

56. $\frac{7}{8} =$

57. $.50 =$

58. $.025 =$

59. $\frac{2}{5} =$

60. $\frac{1}{8} =$

61. $\frac{1}{12} =$

62. $.4 =$

63. $0.375 =$

Write each of the following as a decimal fraction:

64. $\frac{1}{2} =$

65. $\frac{1}{4} =$

66. $\frac{3}{8} =$

67. $40\% =$

68. $4\frac{1}{2}\% =$

69. $\frac{9}{100} =$

70. $\frac{1}{8} =$

71. $\frac{1}{12} =$

72. $12\frac{1}{2}\% =$

73. $6\frac{1}{4}\% =$

Write each of the following as a common fraction:

74. $10\% =$

75. $3\% =$

76. $75\% =$

77. $6\frac{1}{4}\% =$

78. $66\frac{2}{3}\% =$

Complete the following:

79. 25% of 80 =

80. 2.1% of 60 =

81. 125% of 20 =

82. $\frac{1}{4}\%$ of 4000 =

83. ....% of 48 = 16.

84. ....% of 80 = 8.

85. ....% of 20 = 30.

86. ....% of 12 = 32.

87. 4 = ...% of 16.

88. 7 = ...% of 14.

89. 12 = ...% of 60.

90. 6 = ...% of 60.

VI. *General List*

91. What is the average of 4, 6, 8 and 10?

92. Complete: the ratio of 1 to 3 equals the ratio of 2 to

Rewrite the following decimals arranging them in order of their size, the largest first and the smallest last:

93. .93 .15 .94

94. .40 2.5 .875

95. It is necessary for me to mix flour and water in the ratio of 3 to 1. If I use 7½ cups of flour, I must use —— cups of water.

96. On my road map ¼ in. means 8 miles of actual road distance. How many miles apart are two towns which on the map are 1¼ in. apart?

97. Complete: the interest on $150 for 2 months at 6% is

98. What do you pay for goods marked "$8.50 less 2%"?

99. What would you pay for a baseball listed in the catalog as "$2.00 less 40% and 10%"?

100. What per cent of your investment do you make if you invest $300 and gain $6?

Table I presents the per cent of correct responses to each of the preceding tasks. The fifth task is: $\frac{7}{8} + \frac{3}{16} =$. The table shows that of the 622 pupils in the fifth grade approximately one-third (34.6%) were able to do this task; 55.2% of the sixth grade responded correctly; about 3 out of 5 seventh grade pupils did this problem; approximately 2 out of 3 eighth grade pupils succeeded, etc.

TABLE I

PER CENT OF CORRECT RESPONSES OF 3,545 PUPILS IN GRADES 5 TO 12, INCLUSIVE, ON 100 TASKS IN ARITHMETIC

N / Grade / Task	622 / 5	636 / 6	638 / 7	633 / 8	335 / 9	236 / 10	230 / 11	215 / 12
1	85.2	86.1	83.4	91.1	88.7	82.2	89.1	91.2
2	73.4	75.2	81.2	87.2	86.6	86.4	89.6	84.6
3	46.5	47.2	55.2	65.4	67.5	70.0	73.3	82.2
4	57.3	74.4	74.8	77.3	78.8	84.5	85.5	88.8
5	34.6	55.2	58.4	65.5	73.4	74.3	79.9	87.6
6	22.8	42.2	53.1	65.2	71.6	73.0	75.5	82.2
7	24.7	47.3	50.0	60.1	78.2	84.1	84.4	86.6
8	9.2	25.2	39.7	55.3	55.8	65.1	67.2	69.9
9	4.7	25.2	28.5	51.8	60.0	68.8	69.2	71.2
10	1.8	8.1	14.2	31.1	49.6	52.1	61.1	68.8
11	84.4	91.2	91.3	93.2	92.2	96.1	96.8	98.1
12	44.3	59.2	72.9	78.2	84.5	84.7	87.4	89.9
13	25.0	47.2	67.8	71.9	81.5	84.2	88.8	89.8
14	15.2	43.3	45.7	52.2	57.3	61.0	64.4	72.5
15	24.2	48.5	58.6	65.3	65.7	83.0	78.2	81.0
16	30.2	62.5	70.2	82.1	82.1	88.2	89.1	88.2
17	21.4	58.4	66.4	78.2	83.3	85.2	87.7	88.5
18	9.2	32.6	58.9	77.7	79.7	80.5	82.2	92.2
19	0.0	10.1	22.4	51.4	59.1	57.0	61.9	71.4
20	0.8	9.7	18.4	29.4	49.6	57.5	62.7	64.8
21	57.8	62.2	77.8	84.8	78.8	79.0	84.4	91.2
22	52.9	70.7	83.5	89.9	89.6	93.0	93.3	97.7

TABLE I (*Continued*)

N	622	636	638	633	335	236	230	215
Grade	5	6	7	8	9	10	11	12
Task								
23	30.2	40.2	54.5	58.8	69.3	66.2	68.1	72.8
24	13.2	39.2	68.2	81.3	86.6	88.5	91.7	86.6
25	42.4	40.3	75.2	78.8	78.5	82.2	80.0	77.7
26	12.4	36.4	70.2	72.4	73.5	72.0	75.6	72.2
27	14.7	39.5	65.8	74.6	80.1	81.9	83.4	78.8
28	2.1	16.6	30.0	42.2	65.4	53.7	53.2	66.4
29	.5	9.9	31.7	38.6	40.6	38.7	39.4	52.2
30	3.8	10.1	32.1	40.2	52.4	56.0	56.2	73.3
31	18.2	47.4	69.8	76.8	85.6	88.0	92.2	89.7
32	11.4	39.2	53.3	64.0	71.4	77.7	81.4	80.2
33	18.5	46.9	62.2	72.2	74.8	81.0	84.6	84.7
34	8.2	41.8	51.4	60.4	65.5	78.2	81.1	83.5
35	9.3	29.4	46.5	64.5	67.7	83.0	88.2	84.5
36	0.0	1.7	3.1	12.5	16.8	37.7	33.1	41.0
37	0.0	0.2	1.3	8.8	9.2	15.2	21.1	38.2
38	60.1	83.3	88.4	89.2	94.5	93.0	93.3	95.6
39	21.6	29.4	41.5	43.2	50.3	44.0	45.1	57.6
40	22.8	36.3	36.2	48.6	63.8	60.0	63.3	71.0
41	3.5	21.8	32.7	34.8	42.4	52.4	57.6	68.2
42	2.2	19.3	27.0	32.4	44.4	52.2	53.9	55.6
43	3.5	31.2	38.1	40.0	46.5	57.0	58.2	69.5
44	5.1	34.0	39.2	42.6	50.3	58.7	62.4	62.5
45	0.0	9.0	16.1	25.1	40.8	42.2	50.1	62.4
46	0.9	11.2	20.0	29.5	50.5	48.2	52.9	62.0
47	0.8	7.2	15.0	19.9	36.3	30.2	36.4	44.4
48	0.5	1.8	3.4	16.2	27.2	19.9	20.0	27.6
49	9.8	12.2	18.4	19.6	24.8	30.2	39.4	38.5
50	14.4	19.3	22.2	18.0	24.3	31.3	30.0	42.0
51	15.2	14.3	14.4	16.2	23.0	29.5	30.2	41.0
52	4.4	4.4	9.5	17.7	31.6	36.7	38.4	62.2
53	0.0	2.5	3.9	4.2	11.6	12.2	13.0	24.2
54	2.0	9.5	42.0	49.9	56.7	47.5	67.0	81.0
55	5.5	21.2	51.2	82.2	76.3	75.2	89.2	95.2
56	0.6	7.6	35.0	42.1	60.7	47.0	52.0	65.6
57	2.5	7.9	50.0	65.6	76.6	48.8	56.8	65.6
58	0.0	.1	23.1	32.4	38.8	20.8	34.2	52.2
59	6.7	22.1	67.2	73.8	85.4	81.2	83.4	92.6
60	1.8	10.4	54.6	58.8	74.6	46.2	62.4	73.3
61	0.0	5.3	32.4	39.2	41.1	39.7	52.4	56.6
62	0.3	1.6	27.2	38.2	48.2	34.5	44.0	58.3
63	0.0	0.4	27.1	30.0	37.8	28.5	37.4	39.9
64	1.8	20.7	56.2	74.7	79.0	77.7	86.6	90.0
65	0.9	18.8	56.4	71.7	77.5	78.0	81.0	84.6
66	1.2	12.6	48.2	60.2	64.7	70.0	72.4	83.3
67	0.6	7.9	42.6	53.2	51.6	54.7	65.8	74.0
68	0.0	1.0	17.2	24.7	29.5	31.0	36.0	45.9
69	0.0	12.6	31.2	40.0	42.7	45.5	54.0	71.2
70	0.3	4.8	34.8	35.5	44.7	30.5	43.3	50.0
71	0.0	1.5	11.2	16.9	22.5	11.5	22.4	31.6
72	0.0	2.0	25.8	27.8	29.2	24.7	33.0	49.0
73	0.0	0.0	12.6	16.8	21.0	15.5	26.2	38.3
74	1.3	12.3	39.1	48.7	62.4	59.5	63.3	78.3
75	0.8	5.2	25.5	32.8	39.5	32.2	37.6	62.3
76	1.8	14.2	56.4	58.5	63.7	65.7	68.0	76.0

TABLE I (*Continued*)

N Grade Task	622 5	636 6	638 7	633 8	335 9	236 10	230 11	215 12
77	0.0	2.8	24.6	28.4	29.5	22.2	28.0	38.8
78	0.3	5.1	36.4	42.4	48.8	52.2	46.0	62.2
79	0.3	4.6	42.7	50.0	57.5	65.5	62.4	71.1
80	0.0	0.0	4.3	5.8	8.3	9.9	18.5	19.2
81	0.0	1.3	12.6	20.0	31.8	27.7	34.0	44.0
82	0.1	0.0	5.4	5.8	9.2	10.2	12.2	22.5
83	0.0	0.0	9.6	12.8	22.0	16.6	21.6	34.4
84	3.5	10.6	24.9	28.5	39.5	34.7	38.3	44.8
85	0.0	0.0	4.1	8.8	15.6	12.8	18.6	27.4
86	0.0	0.0	1.3	3.8	7.8	7.9	8.4	14.0
87	0.0	0.1	12.3	16.9	24.5	21.2	27.0	38.5
88	0.0	0.2	12.2	18.2	25.2	20.8	26.7	42.2
89	0.0	0.0	8.3	12.7	18.8	17.7	21.5	35.5
90	5.5	9.2	22.1	32.8	40.4	35.5	36.8	48.5
91	9.2	14.7	18.6	39.5	46.5	47.0	51.4	66.2
92	8.3	9.4	24.6	47.2	45.5	45.5	72.2	76.7
93	18.8	31.0	45.6	56.0	64.2	63.2	64.7	71.5
94	0.0	5.3	14.2	36.6	41.2	36.1	48.7	69.0
95	0.8	1.7	12.6	27.5	34.6	34.5	38.5	51.5
96	1.9	6.1	20.1	39.9	44.2	36.1	44.7	58.5
97	0.0	0.3	4.6	20.4	34.6	28.8	41.6	46.8
98	0.0	0.4	12.8	38.5	43.7	30.2	44.7	58.4
99	0.0	0.0	3.6	26.8	24.7	23.3	26.0	34.0
100	0.1	0.2	7.7	22.4	25.7	20.2	34.2	39.5

Table II presents a summary of the results. The table should be interpreted as follows: In the twelfth grade, the last year of the

TABLE II

A SUMMARY SHOWING FOR GRADES 5 TO 12, INCLUSIVE, THE NUMBER OF TASKS FROM 100 TASKS IN ARITHMETIC AT EACH OF VARIOUS LEVELS OF MASTERY

Mastery: Percent of Pupils	5	6	7	8	9	10	11	12
90–100	0	1	1	2	2	3	5	9
80–90	2	2	4	7	11	17	20	20
70–80	1	3	6	15	15	11	7	17
60–70	1	2	7	10	13	8	16	16
50–60	3	3	15	10	10	12	12	10
40–50	3	11	8	12	16	10	8	11
30–40	3	9	14	15	9	17	16	11
20–30	7	8	15	12	14	11	11	4
10–20	10	16	16	13	4	9	4	2
0–10	70	45	14	6	4	2	1	0

senior high school, there were only nine tasks that as many as 90% of the pupils tested could do. There were only 29 tasks that as many as 80% could do. The results for the eighth grade are perhaps of particular interest for in the traditional school this grade marks the completion of formal courses in arithmetic. In this grade there are only two things that nine out of ten children can do; only nine things to which as many as 80% of the children respond correctly. Twenty-four items can be done by as many as seven out of ten children; only 56 items, roughly half of the test, can be done by as many as four out of ten children.

Table III presents certain important results for grades 5 to 12 Perhaps the median score is of greatest interest to the reader. The table reads as follows: one-half of the fifth grade children made a score lower than 17.6; one-half of the sixth grade children

TABLE III

NORMS FOR GRADES 5 TO 12, INCLUSIVE, ON 100 TASKS IN ARITHMETIC

Grade	5	6	7	8	9	10	11	12
No. of Pupils	622	636	638	633	335	236	230	215
75 Percentile	34.2	28.8	50.6	54.2	59.8	62.3	72.2	81.9
Median Score	17.6	22.0	39.8	43.8	48.6	51.7	59.0	67.0
25 Percentile	12.3	17.6	30.4	35.0	39.9	42.6	46.0	54.3
Lowest Score	1.0	3.0	6.0	9.0	5.0	12.0	16.0	22.0
Highest Score	31.0	70.0	86.0	91.0	94.0	92.0	96.0	94.0

made a score lower than 22; one-half of the seventh grade children made a score lower than 39.8, etc. Note the steady rise in these results but do not overlook the fact that improvement from grade to grade is very small in spite of increased maturity, experience in other school subjects, life out of school, and the probability that the weaker students are leaving school.

As a result of this study we know the degree of mastery for 3,545 children on 100 simple tasks in computation. We are able to trace the strength of these mental connections through grades 5 to 12, inclusive.

FINDINGS

1. Although all these processes have been taught in most schools, the degree of mastery at the beginning of the eighth grade is very low and there is little subsequent increase in ability in computation.

2. In spots where increase does appear it may be due to greater maturity or a higher selection of students.

3. Even in the senior high school, mastery is very low. In the tenth grade there are only 51 things out of a 100 to which half the children responded correctly; half of the children know half of the tasks. In the eleventh grade we get 60 tests at the 50% level and in the twelfth grade we get 72 things. That is, at the end of the senior high school half of these children know three-fourths of the material.

4. Further evidence of low mastery is reflected in the median and quartile scores. To be sure, there is a steady march upward in the medians; but it is not so rapid that it may not be due entirely to either maturity, selection, or a combination of these two factors. Note that the median score of the twelfth grade group is only 67; that is to say, one-half of the twelfth-grade pupils individually did less than about half of the test.

5. The highest score agrees with much available evidence of enormous overlapping. For example, while the median score in the twelfth grade is only 67, there is a score as high as 86 in the seventh grade. Recently we found a country boy who did 93 of these tasks.

6. One of the crucial problems of the senior high school, though little is said about it, appears to be that of computation.

7. It does not seem that the formal work in algebra and plane geometry involves enough computation to insure growth. Indeed if maturity and selection are responsible for whatever improvement is shown, it may be that disuse deteriorates the skills.

How well do freshmen in a university know certain fundamentals?

The following table shows the per cent of correct responses given by 2,693 freshmen in the University of Michigan to certain tasks* in mathematics.

TABLE IV

PER CENT OF CORRECT RESPONSES GIVEN BY 2,693 UNIVERSITY FRESHMEN ON A TEST IN FUNDAMENTALS

No.	The test item	Per cent of Correct Responses
	Part I (Arithmetic)	
1.	Divide 5/7 by 6/11	66
2.	What is the 4th power of 3?	65
3.	1/8+5/8= how many hundredths?	67
4.	Multiply 86.5 by 0	67
5.	What is the average of 4, 11, 66, 0, and 9?	71
6.	What is the sum of 7 5/9 and 4 1/2?	78

* The test items used in this study appear in parts 1, 2, and 4 of the Iowa Placement Examinations. The data used in the construction of this table were secured from Dr. C. S. Yoakum in charge of the Office of Educational Investigations.

TABLE IV (*Continued*)

No.	The test item	Per Cent of Correct Responses
7.	Divide 7642.38 by 1000	81
8.	Multiply $2\frac{1}{2}$ by $3\frac{1}{4}$	66
9.	Divide $3\frac{1}{2}$ by $1\frac{3}{4}$	74
10.	Simplify: $\frac{\frac{3}{4}}{\frac{5}{7}}$	63
11.	Reduce 3/20 to a decimal fraction	68
12.	What is the square root of 10609?	39
13.	If 23.186 is multiplied by 7.79, how many digits come after the decimal point?	77
14.	What is the cube of 20.1?	24
15.	Write as a decimal fraction 399 millionths	45
16.	What per cent of 25 is 75?	28
17.	Divide 8/15 by 3.2	34
18.	Simplify: $(1.003)\ (1.3)^3$	22
19.	Simplify: $\frac{3.12}{\sqrt{.04}}$	14
20.	What per cent of 2 is 3/4?	14
	Part II (Algebra)	
1.	If d is the cost per dozen, what is the cost of one?	81
2.	How many factors has $2ab$?	43
3.	Expand $(a-b)^3$	62
4.	What are the factors of $6x^2+7x+2$?	42
5.	What is the square of $(x+y)$?	74
6.	Perform the multiplication: $-6(-2x-4)$	76
7.	Perform the multiplication: $-3x(2+5y)$	64
8.	Perform the multiplication: $(a^2bc^4)^3$	42
9.	Of what binomial is x^2+6x+9 a square?	65
10.	Add: $\frac{1}{a}+\frac{1}{b}=$	27
11.	Write $\sqrt[4]{3mn}$ with an exponent	16
12.	If $cy+d=0$, and c is not equal to 0, what is the value of y?	12
13.	What are the factors of a^2-b^2	74
14.	Simplify: $\frac{27a^6b^4c^2}{3a^2b^2c}$	37
15.	Add: $\frac{1}{m}+n$	19
16.	Add: $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$	24
17.	For what value of x is the expression $\frac{x}{x+1}$ without meaning?	.023

TABLE IV (*Continued*)

No.	The test item	Per Cent of Correct Responses
18.	Simplify: $\sqrt{(64)\ (81)a^2b^{16}}$	14
19.	When $x^2-5x+4=0$ is expressed in the form of $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, how much is added to or subtracted from $-b$?	.068
20.	Write a formula which is equivalent to the following statement: The product of two binomials having a common term equals the square of the common term plus the product of the common term by the sum of the other terms, plus the product of the other terms	.077
	Part III (Presumably a "power" test)	
1.	Write as a power of 12 the number of cubic inches in a cubic foot	34
2.	The perimeter of a square is $4y$. What is the diagonal?	21
3.	The sum of two numbers is 102; the greater exceeds the smaller by 6. What is the greater number?	65
4.	What is the decimal equivalent of the sum, $\frac{7}{10}+\frac{7}{100}+\frac{7}{1000}$?	65
5.	The second of three consecutive even integers is n. What is the first number?	26
6.	What is the difference between the squares of two successive integers, the smaller of which is y?	15
7.	The area of a triangle is ab. What is the product of the base and altitude?	34
8.	Write a formula which gives the cost c in dollars for m miles of wire at d dollars a pound, if one foot of wire weighs p pounds.	20
9.	Given a square of side S, write an equation which states that the perimeter of the square is four times its area	11
10.	N does not equal 0 or 1. If N^{n-1} is 1, what is n?	7
11.	The digits of an integer are a and b. If the product of the digits equals a/b, what is b numerically?	7
12.	If n books cost d dollars, what will x books cost?	20
	For the three problems which follow, write the equations only. Do not solve.	
13.	Find two numbers whose sum is 100 and whose difference is 28.	30
14.	The quotient of two numbers, x and y, is 2, and the larger exceeds the smaller by 7. Find the numbers.	14
15.	The perimeter of a rectangle is 158 inches; the length is four feet more than twice the width. Find the dimensions, x and y, of the rectangle	40

OBSERVATIONS

1. There are only three items in the arithmetic test that as many as 3 out of 4 freshmen can do.
2. There are only 6 items in the algebra section that as many as 1 out of 2 college freshmen can do.
3. There are only 4 items in the problem test that as many as 1 out of 3 can do.

Questions

It is unnecessary to use space formulating conclusions that seem obvious. It may be helpful to state some of the suggested questions that this discussion cannot answer:

1. Does society know what it wants in the way of outcomes from the enormously expensive instruction in the arithmetic and the algebra of our lower schools?
2. Is there any way of finding out what the university expects of entering students? Incidentally, what is a prerequisite?
3. Is it definite enough for a university to say that it requires "a year of algebra" and "a year of geometry"? Or, definitely, what is a year of algebra? and what is a year of geometry?
4. Have these skills merely deteriorated and, if so, how much of a training period would be needed to bring them back to a level at which they would be of some use?
5. Would it be worth while for a university to set aside a brief period for individual diagnosis and remedial work, or is it better to begin new work on the false assumption that students know the fundamentals?
6. What is the passing mark in our schools? (It is commonly assumed to be 60 or 70, actually it is perhaps 30 or 40).
7. The deterioration of mental skills through disuse appears from the evidence in the preceding table to be remarkably fast and widespread. Does this mean that Morrison's argument for "complete mastery" is futile?
8. Are teachers given too much freedom with respect to the choice of materials involving the fundamentals?
9. What kind of a picture would we get if we gave the same test to beginning teachers of mathematics who have "majored" in social studies, physical education, fine arts, Latin, and the like?

While these data raise many questions we cannot answer, they do throw much light on the particular question before us, namely, is there a need for determining a list of specific outcomes? Let us consider this question from another point of view.

Need for Objectives*

The public invests large sums in order to make changes in children with respect to mathematics. Studies of the specific changes being attempted show an astonishing lack of agreement. So great

* Some of the material which follows appeared in the Junior-Senior High School Clearing House, February, 1931.

was the variation shown in Guiler's investigation[1] of thirty courses of study, that he secured his objectives in arithmetic for the elementary grades by accepting each item found in any three courses of study. In 1925 the writer reported an inventory[2] of the content of seven series of mathematics textbooks in the junior-high-school field. For a given grade he found wholly different offerings; e.g., units of arithmetic, algebra, statistics, intuitive and demonstrative geometry and trigonometry. Probably the situation has become more uniform in these grades in recent years, but at that time there was not even agreement concerning the large topics or fields to be covered. In one textbook for a particular grade there were presented, under two topics, discussions of 42 business forms—traveler's check, bill of lading, coupon note, and the like—and yet some of the books for the same grade taught not a single business form. There was also very little agreement with regard to the details—concerning, for example, what specific skills in percentage should be established to, let us say, a ninety per cent mastery.

The greatest variation in mathematics is in the product of teachers with little or with inadequate special preparation. In California and Michigan, for instance, teachers with a secondary-school certificate may, and do teach arithmetic in the elementary schools with no preparation other than their own early work as pupils in these grades. Then, too, from 40 to 60 per cent of the beginning teachers in high school teach subjects which they have never studied in college. During the past six months the writer has visited the schools in a considerable number of college towns where one might expect to find the best samples of the teaching of arithmetic. To be sure, much excellent work was observed; but one reaches the conviction that courses of study are recklessly made, and are far too often carried out by teachers without special preparation and with little or no supervision.

As this sentence is being written two pupils are observed in a struggle with home-work in arithmetic. The fourth-grade pupil is attempting to learn multiplication of whole numbers with the multiplicand and multiplier each having two digits. His thinking is hopelessly confused by the teacher's insistence that he check each problem by a method which was long ago deleted from our better

[1] W. S. Guiler, a doctor's dissertation on file at the University of Chicago.

[2] Raleigh Schorling, *A Tentative List of Objectives in the Teaching of Junior High School Mathematics* (Ann Arbor, Michigan: George Wahr, 1925), vi+140 pages.

textbooks and which, in fact, is far more difficult to learn than is the fundamental process of multiplication. The eighth-grade pupil is struggling with an assignment in which he is expected to discover tests for the divisibility of integers by 2, 3, 6, 8, and 9. There is not the slightest evidence that either the course of study or the teacher* have been affected by the report of the National Committee on Mathematical Requirements. In contrast with other school subjects, arithmetic seems to make a very good showing, however, for 60 to 70 per cent of the items in courses of study are constant. But it is the 30 to 40 per cent of variable elements which cause wasted effort on the part of the teacher and confusion and low mastery on the part of the children. Teachers need and desire that objectives be specific and that they be determined by the best methods now known.

The essence of the scientific method is the examination of all discoverable facts and sources to make sure that we are as nearly right as possible before we go ahead. Only in that way will we achieve a decreasing dependence on rules and traditional practice and an increasing habit of employing the technique of the older sciences.

The Criteria

What specific facts of mathematics shall be taught to the point of mastery to our pupils of the secondary schools? What specific skills and attitudes shall be fixed? What specific concepts shall be developed? The traditional classification of the aims of mathematics (practical, disciplinary, and cultural) is, although valid, so inadequate as to be meaningless as a guide to the teacher in the selection of material. The following eight criteria are proposed as being useful in the choice of the specific changes that are to be made in children.

1. *A Summary of Social Needs.* What mathematics do pupils need to know in order to succeed in the science courses in the high school or in the early years of college? What elements are really important for the next course in mathematics? What items of mathematical knowledge are necessary for the well-informed reader who, considering the enormous growth in the population of the secondary school, now represents perhaps a tenth-grade level?

Beginning studies have been made, of course, in the efforts to answer these questions; but there has been no adequate summary of the unpublished studies now filed in the various libraries, and these studies have been neglected by most of the writers in the field. In fact, in the recent books written for the training of our mathematics

* The situation described does not refer to the writer's home city.

teachers there is almost no trace of the many important and available investigations. It may be argued that such studies include much chaff, but this criticism does not justify the total neglect of the occasional grain. It is to be hoped that the National Council of Teachers of Mathematics will, at the earliest possible date, publish a yearbook that will summarize the important studies in order that the results may be used as a criterion in modifying the courses of study and the textbooks.

2. *Guiding Principles Supplied by a Philosophy of Education.* The road from practised intuition and common sense to the scientific method in education is a continuous one, and philosophy is one of the milestones on it. The philosopher is concerned with values. No item should be, and perhaps none can be, accepted for a curriculum until one has answered such questions as: Is it worth while? and, For whom? The philosopher sees that certain shots sent towards the educational target are wide or short or long. He investigates causes. He may even do a little tabulating. He begins to classify, to arrange, and to organize. He formulates an educational platform of philosophic principles. In spite of the great variation in practice in our secondary schools, certain philosophic principles are receiving wide acceptance and are serving as guides in the shaping of our curricular materials. An illustration of the application of this criterion is furnished in the "Report of the Sub-Committee on Junior High School Mathematics."[3]

3. *Guiding Principles Supplied by Educational Psychology.* Another milestone on the road from common sense to science in education is psychology.When the philosopher becomes interested in how people learn, he looks hopefully to psychology for assistance. As you well know, there has been great activity in the field of educational psychology in recent years. Out of the numerous investigations in the laboratories and classrooms certain principles have, for the moment at least, won general acceptance. No objective should be included in one's list if it is in conflict with a valid principle of educational psychology. The brilliant English schoolmaster, T. Percy Nunn, was among the earliest to apply educational psychology to the selection of mathematical materials; and in recent years mathematics in America has been particularly fortunate in that there have been honest efforts in recent textbooks to apply this criterion.

4. *The Response of the Learners.* Probably the best criterion, though little used, is the pragmatic one of extensive classroom trial. Given a tentative selection—a long list of detailed items—what ones can, in fact, be mastered by children? What ones seem to contribute to growth in mathematical power? To what extent does the pupil accept each item as something worth doing? And, finally, is the task one which he enjoys doing? The criterion amplified by these questions implies that the experimental material needs to be taught by teachers who are interested in learning, who keep systematic records, who are able to draw valid inferences, and who will make periodic reports to the investigator. This test of curricular materials is probably the most valid but teachers need to be critical of claims in prefaces and introductions of books. There have been very few basic experiments involving a detailed record of pupil responses to materials of instruction.

5. *An Inventory of Courses of Study.* It appears that teachers in general neglect courses of study. Recently the writer visited a school which prides itself on having an excellent course of study in arithmetic. Although the course was only a year old, seven of the ten teachers could not locate their copies of the course of study. It is

[3] *North Central Association Quarterly*, II (March, 1928), pp. 9–32.

the writer's opinion, furthermore, that in the whole country there are not as many as ten excellent courses of study in mathematics. Nevertheless, such model and progressive courses as one may find in Denver, Fresno, Chicago, and the State course of study for West Virginia may well serve as instruments to improve the mathematics teaching in our schools. The study by Guiler, to which we have already referred, should be repeated at an early date in order that teachers may have available as a guide a composite of the best practice.

6. *An Inventory of Recent Textbooks.* As has already been stated, the writer in 1925 reported an investigation of seven sets of textbooks in junior-high-school mathematics. This study should now be repeated, and a similar study should be made of textbooks in arithmetic and in each unit of work in the senior high school. These studies should be paralleled by inventories of the corresponding textbooks published ten or twenty years ago. Indeed, survey charts of this type are now in existence as confidential properties of several publishing companies, and it is to be regretted that these valuable documents showing the significant trends in the mathematics curriculum are not generally available to teachers.

7. *The Reports of Standardizing Agencies.* The reports of committees have been very influential in shaping the curriculum. The report of the National Committee on Mathematical Requirements was epoch-making in that it was the first report which represented extensive cooperation in a mobilization of thought with regard to a particular subject. The report was unique, also, in that it attempted to list the specific objectives for the junior high school in greater detail than had ever been done before. Nearly a decade ago the report of this committee was an excellent criterion for the selection of mathematical materials. In all probability the things most worth while in it will persist in our curricula. The converse is unfortunately not true. The report, though still accepted as a guide by many schools, appears to the writer to be in serious need of revision. Ultraconservative communities will find this report a valuable guide for still another decade, but progressive schools are eager to take the next steps in advance.

8. *The Jury System.* In determining objectives, the judgment of competent and experienced teachers is valuable; but it is of the utmost importance that these judges be highly selected, relatively few in number, and entirely willing to undertake the arduous task of passing judgment on each of the many specific items in the curriculum. The value of the outcome of the jury system probably decreases rapidly as the number of judges is increased. There is perhaps no more futile effort in the field of determining objectives than the wholesale collection of opinions of teachers who too frequently give back to instructors of brief courses in summer sessions the materials which these instructors have passed out to their students. If a dozen of the most gifted teachers of mathematics in America could be assembled for a continuous meeting of several months, the outcome would no doubt provide one of the most valuable criteria for the selection of objectives. One of the reasons why the report of the National Committee was so valuable some years ago is that it employed the jury technique. To some it will doubtless appear that the acceptance of this technique negates any claim of a scientific method. To the writer, the judgments of competent teachers will always be an important factor in the scientific method as applied to educational problems.

CONCLUSION

It is the thesis of this discussion that the scientific method requires the use of every available source to make sure that we are as nearly right as possible before we go ahead. Finally, it is believed that when the specific items of information, skills, attitudes, vocabulary, and concepts which constitute the various courses in secondary-school mathematics are checked by the eight criteria here suggested, the outcome is a list of objectives as valid as we at present know how to make them.

A Sample List

It is believed that pupils and parents are entitled to a definite statement of what is expected of the pupils. The partially trained teacher, the beginning teacher, and the teacher trained in other fields — all need "sign posts to indicate the main highways." The experienced and gifted teacher will of course feel free to explore the byways, his initiative will find expression in illustrations and in supplying the side-lights that give meaning to the main body of material in which a high degree of mastery is expected. An illustration of such a list of specific objectives addressed to the pupil of the seventh grade follows.

AN OUTLINE OF ESSENTIALS* IN SEVENTH GRADE MATHEMATICS

Based on the Course of Study—a Guide to Pupils
The University High School, Ann Arbor, Michigan

VOCABULARY

Be Able to Illustrate Each

FIRST SEMESTER

Altitude
Amount of turning
Angle
  Acute
  Corresponding
  Obtuse
  Opposite
  Right
  Supplementary
  Straight
Arc
Average
Budget
Centimeter
Chord
Circle
Compasses
Cross-section Paper
Degree (°)
Diameter
Equation

* This outline is based on two earlier studies: (1) "A Report of the Sub-Committee on Junior High School Mathematics" prepared for the North Central Association of Colleges and Secondary Schools and subsequently published in the March, 1928, issue of the *North Central Association Quarterly*, and (2) *A Tentative List of Objectives in the Teaching of Junior High School Mathematics* (George Wahr, Publisher, Ann Arbor, Michigan). The limitations of space do not permit printing here the outlines for other grades.

Evaluate
Fixed Point
Fraction
  Common
  Decimal
Given Line
Graph
  Bar
  Line
  Circle
Horizontal
Interest
Linear Measure
Meter
Millimeter
Minute (′)
Parallel Lines
Parallelogram
Per Cent
Perpendicular
Plane
Profit
Protractor
Radius
Ratio
Rectangle
Rotation
Rounded Numbers
Scale
Segment
Square
Squared Paper
Standard Unit
Straightedge
Triangle
  Equilateral
  Isoceles
  Right
Unit of Measure
Vertex
Vertical

SECOND SEMESTER

Amount
Area
Balance
Base
Bisect
Bisector
Check
Check Stub
Circumference
Cone
Cube
Cylinder
Diagonal
Discount
Estimate
Exponent
Factor
Formula
Geometric Figures
Hemisphere
Hexagon
Indorse
Interest
Inventory
Octagon
Pay Roll
Perimeter
Perpendicular Bisector
Pi (π)
Price
  Cost
  List
  Selling
Principal
Prism
Profit
  Gross
  Net
Promissory Note
Pyramid
Rate of Interest
Receipts
Regular
Sphere
"To the Nearest Cent"
"To the Nearest Hundredth"
"To the Nearest Per Cent"
"To the Nearest Tenth"
Trapezoid

PRINCIPLES

Be Able to State and Illustrate Each

First Semester

1. Graphs may be used to picture numbers.
2. "Per cent of" means "hundredths times."
3. Common fractions, decimals, and per cents may all mean the same thing.
4. Distances are measured along straight lines, or a straight line is the shortest distance between two points.
5. We show direction by means of angles.
6. A complete rotation is 360°.
7. The sum of the angles of a triangle is 180°.
8. Parallel lines are constructed by making a pair of corresponding angles equal.
9. The perimeter of a rectangle equals twice the altitude plus twice the base.

Second Semester

1. Interest is equal to the principal times the rate times the time.
2. The number of square units in any surface is the area of that surface.
3. The area of a rectangle, a square, or a parallelogram is equal to the base times the height.
4. The area of a triangle is equal to one-half of the product of the base and the height.
5. The area of a circle is equal to π times the square of the radius.
6. The circumference of a circle is equal to π times the diameter.
7. The area of a trapezoid is equal to one-half the height times the sum of the bases.
8. The volume of a rectangular prism is equal to the length times the width times the height.
9. The volume of a cube is equal to the cube of a side.
10. The volume of a cylinder is equal to the area of the base times the height.
11. The volume of a pyramid is equal to one-third the area of the base times the height.
12. The volume of a cone is equal to one-third the area of the base times the height.
13. The surface of a sphere is equal to four times π times the square of the radius.
14. The volume of a sphere is equal to $\frac{4}{3}\pi$ times the cube of the radius.
15. In computing interest on notes, banks use 360 days to a year and 30 days to every month.
16. The radius of the circle, if used as a unit, will "step around" the circle six times.

SKILLS

Be Able To

First Semester

1. Use a table to find desired information.
2. Interpret graphs.
3. Make a neat, accurate bar graph, line graph and circle graph.
4. Find the ratio of one number to another.
5. Recognize equal ratios.
6. Supply the missing part of a ratio when an equal ratio is given.
7. Letter angles so that they may be read easily.
8. Read angles.
9. Recognize opposite, obtuse, acute, and corresponding angles.

10. Find the size of an angle, using a protractor.

11. Draw an angle, using a protractor.

12. Construct, using compasses and straight-edge: (a) a right angle; (b) a square; (c) a rectangle; (d) parallel lines; (e) a parallelogram.

13. In percentage: (a) change common fractions to decimal fractions and per cents; (b) change per cents to common and decimal fractions; (c) find a per cent of a number; (d) find what per cent one number is of another.

14. Read divisions on a ruler.

15. Read lengths of segments by means of cross-section paper.

16. Use metric measure in finding lengths of lines.

17. Change metric units of length to English units and English units to metric units.

18. Express results in "rounded numbers."

19. Use compasses in measuring line segments.

20. Add line segments.

21. Read large numbers, to billions.

SECOND SEMESTER

1. Substitute given values in a formula and compute the result.

2. Understand and work with decimal parts of a per cent.

3. Find the interest when the principal, rate and time are given.

4. Find a discount and net price.

5. Interpret and use exponents.

6. Find the perimeter of a rectangle.

7. Find the circumference of a circle.

8. Find the area of (a) a rectangle; (b) a square; (c) a parallelogram; (d) a triangle; (e) a trapezoid, and (f) a circle.

9. Find the surface of a cylinder and of a sphere.

10. Find the volume of (a) a rectangular prism; (b) a cube; (c) a pyramid; (d) a cylinder; (e) a cone; and (f) a sphere.

11. Bisect: (a) a segment; (b) an angle.

12. Construct by means of a circle: (a) a regular hexagon; (b) an equilateral triangle; (c) a square; (d) a regular octagon.

13. Fill out a check.

14. Indorse a check.

15. Read and interpret promissory notes.

16. Use a payroll to find the wages due.

17. REACH ONE OF THE GOALS IN THE USE OF THE FUNDAMENTAL OPERATIONS AS GIVEN IN OUR PRACTICE BOOKLETS.

FORMULAS

Understand and be Able to Use

FIRST SEMESTER

1. $x+y+z=180°$

SECOND SEMESTER

1. $i=prt$
2. $A=bh$
3. $A=s^2$
4. $A=\frac{bh}{2}$
5. $A=\frac{(B+b)h}{2}$
6. $A=3\frac{1}{7}r^2$
7. $C=3\frac{1}{7}D$
8. $P=2a+2b$

9. $V = lwh$
10. $V = s^3$
11. $V = \frac{Bh}{3}$
12. $V = \pi r^2 h$
13. $V = \frac{\pi r^2 h}{3}$
14. $S = 4\pi r^2$
15. $V = \frac{4}{3}\pi r^3$

MEANINGS (CONCEPTS)

Give at Least Three Illustrations of

FIRST SEMESTER

1. Ratio.
2. Average.
3. Angle.
4. Graphs as pictures of number relations.
5. "Unit of Measure."
6. A line is of indefinite length, a segment is of definite length.
7. A line "a" units long.
8. Direction.
9. Parallel lines.

SECOND SEMESTER

1. Plane figures.
2. Perimeter.
3. Area.
4. Bisecting.
5. Formula.
6. Exponent.
7. Volume.
8. Measurement.

ATTITUDES

Strive For

1. Accuracy.
2. Neatness.
3. High quality of workmanship.
4. Alertness to see results and interpret correctly.
5. Desire for improvement in skills.
6. Habits of thrift.
7. A critical attitude toward reasonableness of results.
8. Interest in growth in skills.
9. Appreciation of geometry in nature, architecture, and art.
10. Precision in statement.
11. An attitude of inquiry—"getting at the bottom of things."
12. Annoyance with vagueness and incompleteness.
13. An appreciation of the value of having suggested solutions to a problem "pass in review" and of evaluating each in turn

The Significance of Specific Lists of Objectives

A list of the detailed items which the pupil is expected to learn at a high level of mastery is of great value alike to pupil, parent and teacher. Although it is not to be expected that such a list of itself will solve the problem of low mastery, it provides a much needed checking device which calls attention sharply to the prime essentials of the course.

The Twelfth Annual Meeting of the National Council of Teachers of Mathematics, Detroit, February 20-21, 1931

Directors Meeting, February 20

Present: John P. Everett, Edwin W. Schreiber, C. M. Austin, Harry C. Barber, William Betz, John R. Clark, Elizabeth Dice, Marie Gugle, J. O. Hassler, Hallie S. Poole, Mary S. Sabin, Vera Sanford, W. S. Schlauch.

The following business was transacted:

Voted that the secretary ask the editor to send copies of past yearbooks to the annual meeting to be placed on sale.

Voted that the president send a telegram of greetings to Professor W. D. Reeve who is on sabbatical leave.

Professor Dunham Jackson of the University of Minnesota presented the final report of the Committee on Geometry and asked that the Board take action similar to that of the Board of Trustees of the Mathematical Association of America, namely, to accept the report, to send a copy of the same to the College Entrance Examination Board, and to discharge the committee.

Voted that the report be accepted and the committee be discharged.

Voted that the secretary send the report of the Geometry Committee to the office of the College Entrance Examination Board together with any suggestions that the Council might have to offer.

Voted that the Council express its sincere appreciation of the services rendered by the Geometry Committee and in particular to Professor Jackson as chairman of that committee.

Voted that the chair appoint a Committee on a Handbook in Mathematics, it being understood that Professor Rankin of Duke University act as chairman.

The Secretary's report of the Eleventh Annual Meeting was approved as read. The Treasurer's report was accepted.

Voted that a committee of three known as the Ballot Committee be appointed with the secretary as chairman and the two most recent ex-presidents as members to prepare a primary ballot suggesting five names for each office and that this primary ballot be published in the November or December issue of THE MATHEMATICS TEACHER.

Voted that the secretary subscribe to various foreign journals of mathematics with the idea that the Council should keep in touch with new movements in the teaching of mathematics in other countries.

Voted that the matter of establishing Honorary Memberships in the Council be taken under consideration and that the secretary take the necessary legal steps to cover Honorary Memberships in the by-laws.

Voted that a Committee on Expansion and Sale of Publications be established consisting of five members including the president and the secretary of the Council.

Voted that there be established within two months of this date a Standing Committee on Cooperation with Official Examining Boards.

Annual Business Meeting, February 21

At this meeting, those who had not previously voted in the annual election

of officers were given an opportunity to do so.

The Secretary-Treasurer's reports were read and accepted.

Voted that the Board of Directors of the National Council take under advisement the matter of experimentation with the proposed one-year course in plane and solid geometry.

The results of the annual election of officers were then announced. A detailed report appears in the April number of THE MATHEMATICS TEACHER.

Directors Luncheon, February 21

The following guests including official delegates of the various mathematics clubs were present:

C. Louis Thiele, Department of Supervision, Detroit, newly elected member of the Board; Professor W. D. Cairns, Secretary-Treasurer of the Mathematical Association of America; Margaret Aylard, Denver; Stanley S. Bash, Columbus; Leolian Carter, St. Joseph; Elizabeth B. Cowley, Pittsburgh; Charles Leckrone, Chicago; Helen H. Marquand, Columbus; A. Brown Miller, Cleveland; Joseph Orleans, New York City; O. E. Overn, Chicago; W. W. Rankin, Duke University; W. O. Smith, Cleveland; Mabel Sykes, Chicago; Florence M. Weston, Minneapolis.

Annual Banquet, February 21

The annual banquet was held at the Detroit-Leland on February 21, upwards of three hundred members and guests being present.

At the conclusion of the dinner, several selections were rendered by the Women's Glee Club of the College of the City of Detroit. President Everett introduced to the members of the Council the new vice-president, Martha Hildebrandt of Maywood, Illinois, and C. Louis Thiele of Detroit, a new member of the Board of Directors. Telegrams were read from Professor W. D. Reeve and from the Mathematics Teachers of the Bay Section of the California State Teachers Association, the latter signifying its wish to become affiliated with the Council. The address of the evening was given by Professor W. D. Cairns of Oberlin, Secretary-Treasurer of the Mathematical Association of America. This address appears elsewhere in this issue. Through the courtesy of the Electrical Research Products, Incorporated, two talking films were presented: "The Play of Imagination in Geometry" by Professor David Eugene Smith, and "Diagnosis of Individual Difficulties in Arithmetic" by Professor Guy Thomas Buswell.

(Signed) EDWIN W. SCHREIBER
Secretary

Registration at the Detroit Meeting February 20-21, 1931

THE NUMBER following the name of a state indicates the total registration from that state. The number following the name of an individual indicates the number of annual meetings of the Council attended by that person according to his registration card.

CALIFORNIA (1): *Oakland,* Gertrude Allen 2.

COLORADO (3): *Denver,* Margaret H.

Aylard, Mary S. Sabin 4; *Greeley,* Arthur E. Mallory.

DISTRICT OF COLUMBIA (3): Caroline Calloway, Ethel H. Grubbs 3, Jennie Wilder 4.

ILLINOIS (27): *Charleston,* E. H. Taylor; *Chicago,* Ethel L. Jarrett, J. T. Johnson 5, Elizabeth Kanies, Charles Leckrone, Orlando Overn, Stafford Simer 4, Charles A. Stone 5, Mabel Sykes 3, Alice Winter 6; *Cicero,* J. Russell McDonald, Walter S. Pope 4, Martha H. Pope; *Evanston,* Arthur Norberg, Mildred Wilson 4; *Macomb,* Vivian Huston 2, Edwin W. Schrieber 8; *Maywood,* Martha Hildebrandt 4, Charlotte Payne; *Normal,* Edith I. Atkin 5, Elinor Flagg 2, C. N. Mills, Bertha Mills; *Oak Park,* C. M. Austin 11, Elsie P. Johnson 4; *Urbana,* E. B. Lytle 5; *Waukegan,* H. G. Ayre.

INDIANA (5): *Culver,* G. H. Crandal; *Indianapolis,* Grace H. Barker; *Muncie,* Christine A. Brown; L. Harper Whitcraft 2; *Terre Haute,* Walter O. Shriner.

IOWA (1): *Cedar Falls,* Dora Kearney 4.

KANSAS (1): *Emporia,* A. W. Phillips 4.

MASSACHUSETTS (2): *Cambridge,* Ralph Beatley 3, Susanne K. Langer.

MICHIGAN (195): *Alma,* Robert W. Clack; *Ann Arbor,* Norman Anning, Inez Bagley, Helen Brown, Gladys Caldwell, Hope Chipman, Lucille Conrad, Heloise Cross, Elizabeth Davis, William H. Dawe, Lucile De Boe, Adelaide Everett, Katherine Hill, Louis P. Jocelyn, Nellie Kahoe, Selma A. Lindell 5, Marion Locke, Olive McLouth, Dorothy Noyes, Susan Potter, Hazel Rich, Raleigh Schorling 11, Isabelle Sewall, Irene Smith, Russell Stabler 2, Ralph Vanderslice; *Battle Creek,* Frances Brandt, Mary Curtis; *Bay City,* E. L. Dersnah, Isabelle Easterly, Meta Ewing, Ann Green, Faye J. Hill, Clara Krause, Grace Payne, Jennie Reagen, R. C. Shellenbarger 2, Irene Washington; *Birmingham,* Lula S. Mergard, Merle Taylor, Emily Walker; *Bloomfield Hills,* Edward Lund, Adolf Nelson; *Dearborn,* Amy Betts, Fred Eshleman, Wm. F. Mullaly, Alice Murray, Norma Reid, Lloyd Shoemaker, Sam Tornberg, Julia M. Williams; *Detroit,* Christian Anderson, N. Reller Barlow, Eleanor Bartnicke, Y. H. Basmadjian, N. E. Bassett, A Rex Carletti, Rose Chesluk, Mabel Colbert, Emmett Corrigan, Mrs. E. Corrigan, Mary Castello, Nina Coughlan, Laurel Crone, Agnes Crow, Dorothy Cullen, Meroe Currey, Wm. Doolittle, Arnold Doub 2, Alvin E. Downer, Geo. P. Edmonds, Ruth Elliott, V. A. Elvers, Martin Fern, Rose Fern, Mary Filch, Ruth Utley Gamber 6, Alice Gordon, Ada Griffin 2, Ruth Haupart, Paula Henze 2, Adelia Hobbs 2, Alice Hurley 2, L. S. Johnston, Inez Jones, A. E. Kelley, Ruth E. King, C. L. Kniffen, B. Larney, C. J. Leonard, Julia Liskow 3, Wm. Littlefield, John Maier 2, Clara Mueller 2, K. MacHale, S. E. McCullough, M. McKenney, Katherine O'Donnell, Ben Pagel, Margaret Pepper, Jean Persons, Verne Philbrick, Duncan Pirie, Richard Poch, R. C. Powell, Lucile Riley, P. N. Sawyer 3, Ruth Saxton, Louise Scholz, H. R. Shrimp, Mary E. Shrimp, Wm. Sinatra, Byron E. Smith, Cecil Soule, Gertrude Steinberger, Sr. Charles Borromeo, Sr. Eustella, Sr. Florence Louise, Sr. Francis Marie, Sr. Jane Edward, Sr. Jean Marie, Sr. Marietta, Sr. Mary Anatolia, Sr. Mary Electa, Sr. M. Felice, Sr. Mary Joela, Sr. Mary Julia, Sr. M. Loyola, Sr. M. Paula, Sr. Mary Paula, Sr. M. Rosanng, Sr. Rose Concepta, Gertrude Teninga, C. Louis Thiele 5, Esther Trainer, Helen

Underhill, Neville Walker, May F. Walsh, Helen Wattles, Henry Wholihan, Orpha Worden 10; *Ferndale,* Olive Maxwell; *Flint,* Florence Bishop 2, Mildred Fitch, Mina Hastings 2, Nellie Loss, Ruth Reik, Vera Shrigley, Blanche Terry, Rose F. Terry, Gretchen Walz; *Fremont,* Marion Nims; *Grand Rapids,* Mabel Inglesh, Dora Reese, Sr. M. Casimir, Sr. M. Veronica, Felix Wilcox; *Howell,* Loron Willis; *Hamtramck,* Kathryn Le Mire; *Highland Park,* Isaac De Voe 2, George Kirkendall, Rachel Kirkendall; *Jackson,* Florence Field, Fred Matthews, Stella Matthews, Harriet M. Mudge, Harriet E. Randolph, Gladdis Richards, Ottelia Sdunek; *Kalamazoo,* Harold Blair, John P. Everett 7, Mrs. J. P. Everett, Ruth A. Sawyer; *Lapeer,* Vera M. Butts, Ruby L. Biggs, Edith Linsley, Alberta Wocholz; *Pontiac,* Ora Hallenbeck, Pearl C. Hollister; *Port Huron,* Isabel Maclaren, Catherine Meehan 2; *Romeo,* Carolyn M. Fisher, J. Foster Moore; *Royal Oak,* Ruth Kirk, Evelyn Schulte, Mary Smith; *Saginaw,* Helen Campbell, Marie Mertz; *Wyandotte,* Elizabeth Blair, Bess Duffey, Alice Johnson, Bess McClintic, Lela Taylor; *Ypsilanti,* John F. Barnhill, Mildred Crawford, Clay Leach.

MINNESOTA (4): *Minneapolis,* Dunham Jackson 3, C. N. Stokes 6, Gertrude Stokes, Florence Weston.

MISSOURI (5): *Clayton,* L. D. Haertter; *Kirksville,* G. H. Jamison 3; *St. Joseph,* Leolian Carter 3; *St. Louis,* Paul Muehlman, Jesse Osborne 2.

NEW HAMPSHIRE (1): *Exeter,* Harry C. Barber 4.

NEW JERSEY (3): *Belleplain,* Fletcher Durell, Thomas J. Durell; *Montclair,* John C. Stone.

NEW YORK (15): *Buffalo,* Mary E. Crafts 3, Hallie S. Poole 3, Margaret Wilkins; *Jamestown,* Maude A. La Vier 2; *New York City,* John R. Clark 11, Joseph McCormack 3, Joseph B. Orleans 2, William S. Schlauch 4, Helen M. Walker; *Rochester,* William Betz 7, Louis Hagen, H. Carlisle Taylor, Paul Welton; *Syracuse,* Floyd F. Decker; *Yonkers,* Arthur Otis 6.

NORTH CAROLINA (1): *Durham,* W. W. Rankin 4.

OHIO (33): *Akron,* Gaylord Montgomery; *Athens,* H. E. Benz 3; *Bowling Green,* Anna N. Gryting, Lena Mills, J. R. Overman 6; *Cincinnati,* Marie Becker 2, Norma Struke 2; *Cleveland,* Eleanor Applegate 2, Fred Burroughs 3, Cora Lederer, E. Michalske, A. Brown Miller 5, Vera Sanford 5, W. O. Smith 2, Maude Tomlin; *Columbus,* Stanley S. Bash, Maude Davis, Marie Gugle 12, Edna B. Hatton, C. B. Marquand 7, Helen H. Marquand 7, Frances Mitzenberg 5, Mary Moler 2, Amy F. Preston 9; *Findlay,* Glendora Mills; *Lorain,* Helen Simpson 5; *Maple Heights,* Helen Jatzeh, Bernice Wiltshire 2; *Oberlin,* W. D. Cairns 3; *Oxford,* Carmille Holley; *Ravenna,* Hazel E. Collins 2; *Toledo,* Elizabeth Johnson, R. M. Sprague 3.

OKLAHOMA (2): *Norman,* J. O. Hassler 4, *Oklahoma City,* Martha Denny.

PENNSYLVANIA (1): *Pittsburgh,* Elizabeth Cowley 2.

TENNESSEE (1): *Nashville,* Frank L. Wren.

TEXAS (3): *Dallas,* Elizabeth Dice 2; *Houston,* Bertha L. Doering, Paula C. Doering.

WISCONSIN (1): *Racine,* Mary Potter 8.

TOTAL: 308.

Report of the Secretary-Treasurer as Treasurer
February 1, 1931

Receipts

Balance Feb. 1, 1930		$ 91.44
W. D. Reeve, MATHEMATICS TEACHER	$1,224.20	
W. D. Reeve, MATHEMATICS TEACHER	1,158.21	
W. D. Reeve, Year Books	306.25	
W. D. Reeve, Advertizing	42.00	
Bureau of Publications Year Books	845.47	
Cash from Cert. of Deposit	800.00	
Cash from Detroit Ed. Bond	1,067.50	
Cash from Savings Acc.	100.00	
Math. Ass'n of America	51.86	
		5,595.49
Total Receipts		$5,686.93

Other Assets:

New York Telephone Bond	$1,065.00
Savings	174.37
Total	$1,239.37

Expenditures

Annual Meeting:		
Director's Expenses	$ 356.13	
Program Speakers	96.82	
Local Committee	57.87	
Printing	14.08	
		$ 524.90
President's Office:		
Traveling Expense, Stamps, etc.		48.43
Editor's Office:		
Printing of Register	1,090.37	
Printing of Index	605.07	
Typing, clerical work	92.50	
		1,787.94
Geometry Committee:		
Expenses		103.72
Secretary-Treasurer's Office:		
Gazette Press (1925)	1,193.53	
Savings Acc. Loan	101.00	
Stationery, Supplies	53.71	
Clerical work	16.00	
Recording Fee	1.30	
		1,365.54
Total Expenditures		3,830.53
Cash on Hand		1,856.40
		$5,686.93

EDWIN W. SCHREIBER
Treasurer